Mirrors and Reflections:
The Geometry of Finite Reflection Groups

Incomplete Draft Version 01

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25 February 2000

Introduction

This expository text contains an elementary treatment of finite groups generated by reflections. There are many splendid books on this subject, particularly [**H**] provides an excellent introduction into the theory. The only reason why we decided to write another text is that some of the applications of the theory of reflection groups and Coxeter groups are almost entirely based on very elementary geometric considerations in Coxeter complexes. The underlying ideas of these proofs can be presented by simple drawings much better than by a dry verbal exposition. Probably for the reason of their extreme simplicity these elementary arguments are mentioned in most books only briefly and tangently.

We wish to emphasize the intuitive elementary geometric aspects of the theory of reflection groups. We hope that our approach allows an easy access of a novice mathematician to the theory of reflection groups. This aspect of the book makes it close to [**GB**]. We realise, however, that, since classical Geometry has almost completely disappeared from the schools' and Universities' curricula, we need to smugle it back and provide the student reader with a modicum of Euclidean geometry and theory of convex polyhedra. We do not wish to appeal to the reader's geometric intuition without trying first to help him or her to develope it. In particular, we decided to saturate the book with visual material. Our sketches and diagrams are very unsophisticated; one reason for this is that we lack skills and time to make the pictures more intricate and aesthetically pleasing, another is that the book was tested in a M. Sc. lecture course at UMIST in Spring 1997, and most pictures, in their even less sophisticated versions, were first drawn on the blackboard. There was no point in drawing pictures which could not be reproduced by students and reused in their homework. Pictures are not for decoration, they are indispensable (though maybe greasy and soiled) tools of the trade.

The reader will easily notice that we prefer to work with the mirrors of reflections rather than roots. This approach is well known and fully exploited in Chapter 5, §3 of Bourbaki's classical text [**Bou**]. We have combined it with Tits' theory of chamber complexes [**T**] and thus made the exposition of the theory entirely geometrical.

The book contains a lot of exercises of different level of difficulty. Some of them may look irrelevant to the subject of the book and are included for the sole purpose of developing the geometric intuition of a student. The more experienced reader may skip most or all exercises.

Prerequisites

Formal prerequisites for reading this book are very modest. We assume the reader's solid knowledge of Linear Algebra, especially the theory of orthogonal transformations in real Euclidean spaces. We also assume that they are familiar with the following basic notions of Group Theory:

> groups; the order of a finite group; subgroups; normal subgroups and factorgroups; homomorphisms and isomorphisms; permutations, standard notations for them and rules of their multiplication; cyclic groups; action of a group on a set.

You can find this material in any introductory text on the subject. We highly recommend a splendid book by M. A. Armstrong [**A**] for the first reading.

Acknowledgements

The early versions of the text were carefully read by Robert Sandling and Richard Booth who suggested many corrections and improvements.

Our special thanks are due to the students in the lecture course at UMIST in 1997 where the first author tested this book:

> Bo Ahn, Ayşe Berkman, Richard Booth, Nazia Kalsoom, Vaddna Nuth.

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1.1 Affine Euclidean space $\mathbb{A}R^n$

1.1.1 How to read this section

This section provides only a very sketchy description of the affine geometry and can be skipped if the reader is familiar with this standard chapter of Linear Algebra; otherwise it would make a good exercise to restore the proofs which are only indicated in our text[1]. Notice that the section contains nothing new in comparision with most standard courses of Analytic Geometry. We simply transfer to n dimensions familiar concepts of three dimensional geometry.

The reader who wishes to understand the rest of the course can rely on his or her three dimensional geometric intuition. The theory of reflection groups and associated geometric objects, root systems, has the most fortunate property that almost all computations and considerations can be reduced to two and three dimensional configurations. We shall make every effort to emphasise this intuitive geometric aspect of the theory. But, as a warning to students, we wish to remind you that our intuition would work only when supported by our ability to prove rigorously 'intuitively evident' facts.

[1]To attention of students: the material of this section will not be included in the examination.

1.1.2 Euclidean space $\mathbb{R}^n$

Let $\mathbb{R}^n$ be the Euclidean n-dimensional real vector space with canonical scalar product $(\ ,\ )$. We identify $\mathbb{R}^n$ with the set of all column vectors

$$\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

of length n over $\mathbb{R}$, with componentwise addition and multiplication by scalars, and the scalar product

$$\begin{aligned} (\alpha, \beta) = \alpha^t \beta &= (a_1, \dots, a_n) \begin{pmatrix} b_1 \\ \vdots \\ a_n \end{pmatrix} \\ &= a_1 b_1 + \cdots + a_n b_n; \end{aligned}$$

here ^{t} denotes taking the transposed matrix.

This means that we fix the canonical orthonormal basis $\epsilon_1, \dots, \epsilon_n$ in $\mathbb{R}^n$, where

$$\epsilon_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \quad (\text{ the entry 1 is in the } i\text{th row}) .$$

The *length* $|\alpha|$ of a vector α is defined as $|\alpha| = \sqrt{(a, a)}$. The *angle* A between two vectors α and β is defined by the formula

$$\cos A = \frac{(\alpha, \beta)}{|\alpha||\beta|}, \quad 0 \leqslant A < \pi.$$

If $\alpha \in \mathbb{R}^n$, then

$$\alpha^\perp = \{ \beta \in \mathbb{R}^n \mid (\alpha, \beta) = 0 \}$$

in the linear subspace *normal* to α. If $\alpha \neq 0$ then $\dim \alpha^\perp = n - 1$.

1.1.3 Affine Euclidean space $\mathbb{A}R^n$

The real affine Euclidean space $\mathbb{A}R^n$ is simply the set of all n-tuples $a_1, \dots, a_n$ of real numbers; we call them *points*. If $a = (a_1, \dots, a_n)$ and

$b = (b_1, \dots, b_n)$ are two points, the *distance* $r(a, b)$ between them is defined by the formula

$$r(a, b) = \sqrt{(a_1 - b_1)^2 + \cdots + (a_n - b_n)^2}.$$

On of the most basic and standard facts in Mathematics states that this distance satisfies the usual axioms for a metric: for all $a, b, c \in \mathbb{R}^n$,

- $r(a, b) \geqslant 0$;
- $r(a, b) = 0$ if and only if $a = b$;
- $r(a, b) + r(b, c) \geqslant r(a, c)$ (the Triangle Inequality).

With any two points a and b we can associate a vector[2] in $\mathbb{R}^n$

$$\vec{ab} = \begin{pmatrix} b_1 - a_1 \\ \vdots \\ b_n - a_n \end{pmatrix}.$$

If a is a point and α a vector, $a + \alpha$ denotes the unique point b such that $\vec{ab} = \alpha$. The point a will be called the *initial*, b the *terminal* point of the vector $\vec{ab}$. Notice that

$$r(a, b) = |\vec{ab}|.$$

The real Euclidean space $\mathbb{R}^n$ models what physicists call the system of *free vectors*, i.e. physical quantities characterised by their magnitude and direction, but whose application point is of no consequence. The *n-dimensional affine Euclidean space* $\mathbb{A}R^n$ is a mathematical model of the system of *bound* vectors, that is, vectors having fixed points of application.

1.1.4 Affine subspaces

Subspaces. If U is a vector subspace in $\mathbb{R}^n$ and a is a point in $\mathbb{A}R^n$ then the set

$$a + U = \{\, a + \beta \mid \beta \in U \,\}$$

is called an *affine subspace* in $\mathbb{A}R^n$. The *dimension* $\dim A$ of the affine subspace $A = a + U$ is the dimension of the vector space U. The *codimension* of an affine subspace A is $n - \dim A$.

[2] It looks a bit awkward that we arrange the coordinates of points in rows, and the coordinates of vectors in columns. The row notation is more convenient typographically, but, since we use left notation for group actions, we have to use column vectors: if A is a square matrix and α a vector, the notation $A\alpha$ for the product of A and α requires α to be a column vector.

If A is an affine subspace and $a \in A$ a point then the set of vectors

$$\vec{A} = \{ \vec{ab} \mid b \in A \}$$

is a vector subspace in $\mathbb{R}^n$; it coincides with the set

$$\{ \vec{bc} \mid b, c \in A \}$$

and thus does not depend on choice of the point $a \in A$. We shall call $\vec{A}$ the *vector space* of A. Notice that $A = a + \vec{A}$ for any point $a \in A$. Two affine subspaces A and B of the same dimension are *parallel* if $\vec{A} = \vec{B}$.

Systems of linear equations. The standard theory of systems of simultaneous linear equaitions characterises affine subspaces as solution sets of systems of linear equations

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= c_1 \\ a_{21}x_1 + \cdots + a_{2n}x_n &= c_2 \\ \vdots \quad &\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= c_m. \end{aligned}$$

In particular, the intersection of affine subspaces is either an affine subspace or the empty set. The codimension of the subspace given by the system of linear equations is the maximal number of linearly independent equations in the system.

Points. Points in $\mathbb{A}R^n$ are 0-dimensional affine subspaces.

Lines. Affine subspaces of dimension 1 are called *straight lines* or *lines*. They have the form

$$a + \mathbb{R}\alpha = \{ a + t\alpha \mid t \in \mathbb{R} \},$$

where a is a point and α a non-zero vector. For any two distinct points $a, b \in \mathbb{A}R^n$ there is a unique line passing through them, that is, $a + \mathbb{R}\vec{ab}$. The *segment* $[a, b]$ is the set

$$[a, b] = \{ a + t\vec{ab} \mid 0 \leqslant t \leqslant 1 \},$$

the *interval* (a, b) is the set

$$(a, b) = \{ a + t\vec{ab} \mid 0 < t < 1 \}.$$

Planes. Two dimensional affine subspaces are called *planes*. If three points a, b, c are not *collinear*, i.e. do not belong to a line, then there is a unique plane containing them, namely, the plane

$$a + \mathbb{R}\vec{ab} + \mathbb{R}\vec{ac} = \{\, a + u\vec{ab} + v\vec{ac} \mid u, v, \in \mathbb{R} \,\}.$$

A plane contains, for any its two distinct points, the entire line connecting them.

Hyperplanes, that is, affine subspaces of codimension 1, are given by equations

$$a_1x_1 + \cdots + a_nx_n = c. \tag{1.1}$$

If we represent the hyperplane in the vector form $b + U$, where U is a $(n-1)$-dimensional vector subspace of $\mathbb{R}^n$, then $U = \alpha^\perp$, where

$$\alpha = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}.$$

Two hyperplanes are either parallel or intersect along an affine subspace of dimension $n-2$.

1.1.5 Half spaces

If H is a hyperplane given by Equation 1.1 and we denote by $f(x)$ the linear function

$$f(x) = a_1x_1 + \cdots + a_nx_n - c,$$

where $x = (x_1, \dots, x_n)$, then the hyperplane divides the affine space $\mathbb{A}R^n$ in two *open half spaces* V^+ and V^- defined by the inequalities $f(x) > 0$ and $f(x) < 0$. The sets $\overline{V}^+$ and $\overline{V}^-$ defined by the inequalities $f(x) \geqslant 0$ and $f(x) \leqslant 0$ are called *closed half spaces*. The half spaces are *convex* in the following sense: if two points a and b belong to one half space, say, V^+ then the restriction of f onto the segment

$$[a, b] = \{\, a + t\vec{ab} \mid 0 \leqslant t \leqslant 1 \,\}$$

is a linear function of t which cannot take the value 0 on the segment $0 \leqslant t \leqslant 1$. Hence, with any its two points a and b, a half space contains the segment $[a, b]$. Subsets in $\mathbb{A}R^n$ with this property are called *convex*.

More generally, a *curve* is an image of the segment $[0, 1]$ of the real line $\mathbb{R}$ under a continuous map from $[0, 1]$ to $\mathbb{A}R^n$. In particular, a segment $[a, b]$ is a curve, the map being $t \mapsto a + t\vec{ab}$.

Two points a and b of a subset $X \subseteq \mathbb{A}R^n$ are *connected* in X if there is a curve in X containing both a and b. This is an equivalence relation, and its classes are called *connected components* of X. A subset X is *connected* if it consists of just one connected component, that is, any two points in X can be connected by a curve belonging to X. Notice that any convex set is connected; in particular, half spaces are connected.

If H is a hyperplane in $\mathbb{A}R^n$ then its two open halfspaces V^- and V^+ are connected components of $\mathbb{A}R^n \setminus H$. Indeed, the halfspaces V^+ and V^- are connected. But if we take two points $a \in V^+$ and $b \in V^-$ and consider a curve

$$\{\, x(t) \mid t \in [0,1]\,\} \subset \mathbb{A}R^n$$

connecting $a = x(0)$ and $b = x(1)$, then the continuous function $f(x(t))$ takes the values of opposite sign at the ends of the segment $[0,1]$ and thus should take the value 0 at some point t_0, $0 < t_0 < 1$. But then the point $x(t_0)$ of the curve belongs to the hyperplane H.

1.1.6 Bases and coordinates

Let A be an affine subspace in $\mathbb{A}R^n$ and $\dim A = k$. If $o \in A$ is an arbitrary point and $\alpha_1, \ldots, \alpha_k$ is an orthonormal basis in $\vec{A}$ then we can assign to any point $a \in A$ the coordinates $(a_1, \ldots, a_k)$ defined by the rule

$$a_i = (\vec{oa}, \alpha_i), \quad i = 1, \ldots, k.$$

This turns A into an affine Euclidean space of dimension k which can be identified with $\mathbb{A}R^k$. Therefore everything that we said about $\mathbb{A}R^n$ can be applied to any affine subspace of $\mathbb{A}R^n$.

We shall use change of coordinates in the proof of the following simple fact.

Proposition 1.1.1 *Let a and b be two distinct points in $\mathbb{A}R^n$. The set of all points x equidistant from a and b, i.e. such that $r(a,x) = r(b,x)$ is a hyperplane normal to the segment $[a,b]$ and passing through its midpoint.*

Proof. Take the midpoint o of the segment $[a,b]$ for the origin of an orthonormal coordinate system in $\mathbb{A}R^n$, then the points a and b are represented by the vectors $\vec{oa} = \alpha$ and $\vec{ob} = -\alpha$. If x is a point with $r(a,x) = r(b,x)$ then we have, for the vector $\chi = \vec{ox}$,

$$\begin{aligned} |\chi - \alpha| &= |\chi + \alpha|, \\ (\chi - \alpha, \chi - \alpha) &= (\chi + \alpha, \chi + \alpha), \\ (\chi, \chi) - 2(\chi, \alpha) + (\alpha, \alpha) &= (\chi, \chi) + 2(\chi, \alpha) + (\alpha, \alpha), \end{aligned}$$

which gives us

$$(\chi, \alpha) = 0.$$

But this is the equation of the hyperplane normal to the vector α directed along the segment $[a, b]$. Obviously the hyperplane contains the midpoint o of the segment. □

1.1.7 Convex sets

Recall that a subset $X \subseteq \mathbb{A}R^n$ is *convex* if it contains, with any points $x, y \in X$, the segment $[x, y]$ (Figure 1.7).

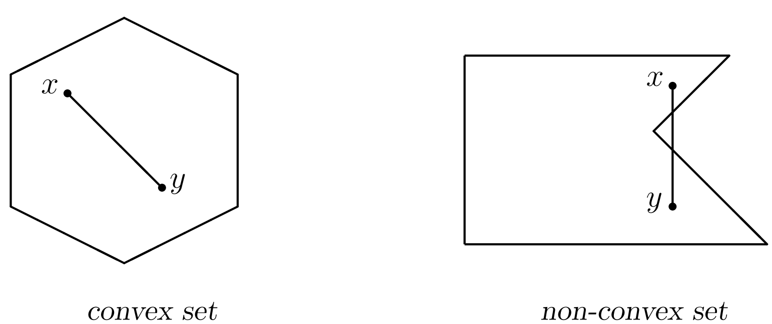


Figure 1.1: Convex and non-convex sets.

Obviously the intersection of a collection of convex sets is convex. Every convex set is connected. Affine subspaces (in particular, hyperplanes) and half spaces in $\mathbb{A}R^n$ are convex. If a set X is convex then so are its closure $\overline{X}$ and interior X°. If $Y \subseteq \mathbb{A}R^n$ is a subset, it *convex hull* is defined as the intersection of all convex sets containing it; it is the smallest convex set containing Y.

Exercises

1.1.1 Prove that the complement to a 1-dimensional linear subspace in the 2-dimensional complex vector space $\mathbb{C}^2$ is connected.

1.1.2 In a well known textbook on Geometry [**Ber**] the affine Euclidean spaces are defined as triples $(A, \vec{A}, \Phi)$, where $\vec{A}$ is an Euclidean vector space, A a set and Φ a faithful simply transitive action of the additive group of $\vec{A}$ on A [**Ber**, vol. 1, pp. 55 and 241]. Try to understand why this is the same object as the one we discussed in this section.

1.2 Hyperplane arrangements

This section follows the classical treatment of the subject by Bourbaki [**Bou**], with slight changes in terminology. All the results mentioned in this section are intuitively self-evident, at least after drawing a few simple pictures. We omit some of the proofs which can be found in [**Bou**, Chap. V, §1].

1.2.1 Chambers of a hyperplane arrangement

A finite set Σ of hyperplanes in $\mathbb{A}R^n$ is called a *hyperplane arrangement.* We shall call hyperplanes in Σ *walls* of Σ.

Given an arrangement Σ, the hyperlanes in Σ cut the space $\mathbb{A}R^n$ and each other in pieces called faces, see the explicit definition below. We wish to develop a terminology for the description of relative position of faces with respect to each other.

If H is a hyperplane in $\mathbb{A}R^n$, we say that two points a and b of $\mathbb{A}R^n$ are on the *same side* of H if both of them belong to one and the same of two halfspaces V^+, V^- determined by H; a and b are *similarly positioned* with respect to H if both of them belong simultaneously to either V^+, H or V^-.

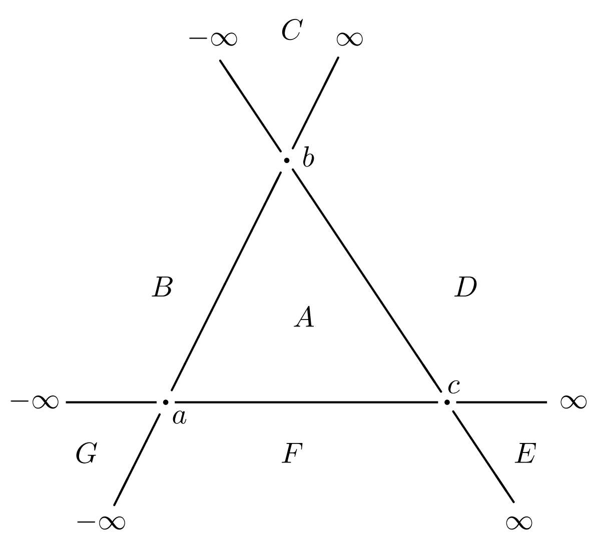


Figure 1.2: *Three lines in general position* (*i.e. no two lines are parallel and three lines do not intersect in one point*) *divide the plane into seven open faces* $A, \dots, G$ (*chambers*), *nine* 1*-dimensional faces* (*edges*) $(-\infty, a), (a, b), \dots, (c, \infty)$, *and three* 0*-dimensional faces* (*vertices*) a, b, c. *Notice that* 1*-dimensional faces are open intervals.*

Let Σ be a finite set of hyperplanes in $\mathbb{A}R^n$. If a and b are points in $\mathbb{A}R^n$, we shall say that a and b are *similarly positioned* with respect to Σ and write $a \sim b$ if a and b are similarly positioned with respect to every hyperplane $H \in \Sigma$. Obviously $\sim$ is an equivalence relation. Its equivalence classes are called *faces* of the hyperplane arrangement Σ (Figure 1.2). Since Σ is finite, it has only finitely many faces. We emphasise that faces are *disjoint*; distinct faces have no points in common.

It easily follows from the definition that if F is a face and a hyperplane $H \in \Sigma$ contains a point in F then H contains F. The intersection L of all hyperplanes in Σ which contain F is an affine subspace, it is called the *support* of F. The *dimension* of F is the dimension of its support L.

Topological properties of faces are described by the following result.

Proposition 1.2.1 *In this notation,*

- *F is an open convex subset of the affine space L.*
- *The boundary of F is the union of some set of faces of strictly smaller dimension.*
- *If F and F' are faces with equal closures, $\overline{F} = \overline{F'}$, then $F = F'$.*

Chambers. By definition, *chambers* are faces of Σ which are not contained in any hyperplane of Σ. Also chambers can be defined, in an equivalent way, as connected components of

$$\mathbb{A}R^n \smallsetminus \bigcup_{H \in \Sigma} H.$$

Chambers are open convex subsets of $\mathbb{A}R^n$. A *panel* or *facet* of a chamber C is a face of dimension $n-1$ on the boundary of C. It follows from the definition that a panel P belongs to a unique hyperplane $H \in \Sigma$, called a *wall* of the chamber C.

Proposition 1.2.2 *Let C and C' be two chambers. The following conditions are equivalent:*

- *C and C' are separated by just one hyperplane in Σ.*
- *C and C' have a panel in common.*
- *C and C' have a unique panel in common.*

Lemma 1.2.3 *Let C and C' be distinct chambers and P their common panel. Then*

(a) *the wall H which contains P is the only wall with a notrivial intersection with the set $C \cup P \cup C'$, and*

(b) *$C \cup P \cup C'$ is a convex open set.*

Proof. The set $C \cup P \cup C'$ is a connected component of what is left after deleting from V all hyperplanes from Σ but H. Therefore H is the only wall in σ which intersects $C \cup P \cup C'$. Moreover, $C \cup P \cup C'$ is the intersection of open half-spaces and hence is convex. □

1.2.2 Galleries

We say that chambers C and C' are *adjacent* if they have a panel in common. Notice that a chamber is adjacent to itself. A *gallery* Γ is a sequence $C_0, C_1, \dots, C_l$ of chambers such that C_i and C_{i-1} are adjacent, for all $i = 1, \dots, l$. The number l is called the *length* of the gallery. We say that C_0 and C_l are *connected* by the gallery Γ and that C_0 and C_l are the *endpoints* of Γ. A gallery is *geodesic* if it has the minimal length among all galleries connecting its endpoints. The *distance* $d(C, D)$ between the chambers C and D is the length of a geodesic gallery connecting them.

Proposition 1.2.4 *Any two chambers of Σ can be connected by a gallery. The distance $d(D, C)$ between the chambers C and D equals to the number of hyperplanes in Σ which separate C from D.*

Proof. Assume that C and D are separated by m hyperplanes in Σ. Select two points $c \in C$ and $d \in D$ so that the segment $[c, d]$ does not intersect any $(n-2)$-dimensional face of Σ. Then the chambers which are intersected by the segment $[c, d,]$ form a gallery connecting C and D, and it is easy to see that its length is m. To prove that $m = d(C, D)$, consider an arbitrary gallery $C_0, \dots, C_l$ connecting $C = C_0$ and $D = C_l$. We may assume without loss of generality that consequent chambers C_{i-1} and C_i are distinct for all $i = 1, \dots, l$. For each $i = 0, 1, \dots, l$, chose a point $c_i \in C_i$. The union

$$[c_0, c_1] \cup [c_1, c_2] \cup \cdots \cup [c_{l-1}, c_l]$$

is connected, and by the connectedness argument each wall H which separates C and D has to intersect one of the segments $[c_{i-1}, c_i]$. Let P be the common panel of C_{i-1} and C_i. By virtue of Lemma 1.2.3(a), $[c_{i-1}, c_i] \subset C_{i-1} \cup P \cup C_i$ and H has a nontrivial intersection with $C_{i-1} \cup P \cup C_i$. But then, in view of Lemma 1.2.3(b), H contains the panel P. Therefore each of m walls separating C from D contains the common panel of a different pair (C_{i-1}, C_i) of adjacent chambers. It is obvious now that $l \geqslant m$. □

As a byproduct of this proof, we have another useful result.

Lemma 1.2.5 *Assume that the endpoints ot the gallery $C_0, C_1, \ldots, C_l$ lie on the opposite sides of the wall H. Then, for some $i = 1, \ldots, l$, the wall H contains the common panel of consequtive chambers C_{i-1} and C_i.*

We shall say in this situation that the wall H *interesects* the gallery $C_0, \ldots, C_l$.

Another corollary of Proposition 1.2.4 is the following characterisation of geodesic galleries.

Proposition 1.2.6 *A geodesic gallery intersects each wall at most once.*

The following elementary property of distance $d(\ ,\ )$ will be very useful in the sequel.

Proposition 1.2.7 *Let D and E be two distinct adjacent chambers and H wall separating them. Let C be a chamber, and assume that the chambers C and D lie on the same side of H. Then*

$$d(C, E) = d(C, D) + 1.$$

Proof is left to the reader as an exercise. □

Exercises

1.2.1 *Prove that distance $d(\ ,\ )$ on the set of chambers of a hyperplane arrangement satisfies the triangle inequality:*

$$d(C, D) + d(C, E) \geqslant d(C, E).$$

1.2.2 Prove that, in the plane $\mathbb{A}R^2$, n lines in general position (i.e. no lines are parallel and no three intersect in one point) divide the plane in

$$1 + (1 + 2 + \cdots + n) = \frac{1}{2}(n^2 + n + 2)$$

chambers. How many of these chambers are unbounded? Also, find the numbers of 1- and 0-dimensional faces.

Hint: Use induction on n.

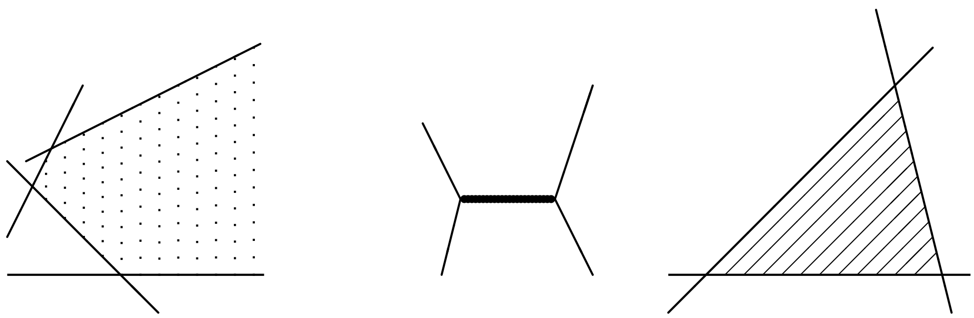
1.2.3 Given a line arrangement in the plane, prove that the chambers can be coloured black and white so that adjacent chambers have different colours.

Hint: Use induction on the number of lines.

1.2.4 Prove Proposition 1.2.7.

Hint: Use Proposition 1.2.4 *and Lemma* 1.2.3.

1.3 Polyhedra



(a) (b) (c)

Figure 1.3: *Polyhedra can be unbounded* (a) *or without interior points* (b)*. In some books the term 'polytope' is reserved for bounded polyhedra with interior points* (c)*; we prefer to use it for all bounded polyhedra, so that* (b) *is a polytope in our sense.*

A *polyhedral set*, or *polyhedron* in $\mathbb{A}R^n$ is the intersection of the finite number of closed half spaces. Since half spaces are convex, every polyhedron is convex. Bounded polyhedra are called *polytopes* (Figure 1.3).

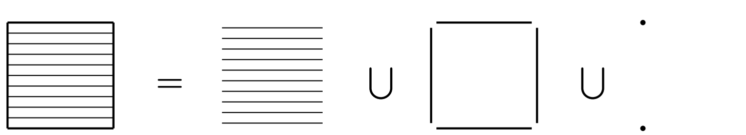


Figure 1.4: *A polyhedron is the union of its faces.*

Let Δ be a polyhedron represented as the intersection of closed halfspaces $X_1, \dots, X_m$ bounded by the hyperplanes $H_1, \dots, H_m$. Consider the hyperplane configuration $\Sigma = \{ H_1, \dots, H_m \}$. If F is a face of Σ and has a point in common with Δ then F belongs to Δ. Thus Δ is a union of faces. Actually it can be shown that Δ is the closure of exactly one face of Σ.

0-dimensional faces of Δ are called *vertices*, 1-dimensional *edges*.

The following result is probably the most important theorem about polytopes.

Theorem 1.3.1 *A polytope is the convex hull of its vertices. Vice versa, given a finite set E of points in $\mathbb{A}R^n$, their convex hull is a polytope whose vertices belong to E.*

As R. T. Rockafellar characterised it [**Roc**, p. 171],

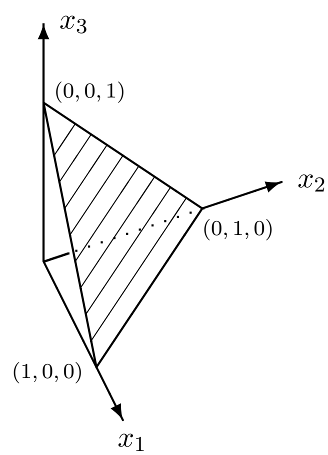
> This classical result is an outstanding example of a fact which is a completely obvious to geometric intuition, but which wields important algebraic content and not trivial to prove.

We hope this quotation is a sufficient justification for our decision not include the proof of the theorem in our book.

Exercises

1.3.1 Let Δ be a tetrahedron in $\mathbb{A}R^3$ and Σ the arrangement formed by the planes containing facets of Δ. Make a sketch analogous to Figure 1.2. Find the number of chambers of Σ. Can you see a natural correspondence between chambers of Σ and faces of Δ?

Hint: When answering the second question, consider first the 2*-dimensional case, Figure* 1.2.



The regular 2*-simplex is the set of solutions of the system of simultaneous inequalities and equation*

$$x_1 + x_2 + x_3 = 0,$$

$$x_1 \geqslant 0,\ x_2 \geqslant 0,\ x_3 \geqslant 0.$$

We see that it is an equilateral triangle.

Figure 1.5: The regular 2-simplex

1.3.2 The previous exercise can be generalised to the case of n dimensions in the following way. By definition, the *regular n-simplex* is the set of solutions of the system of simultaneous inequalities and equation

$$\begin{array}{rcl} x_1 + \cdots + x_n + x_{n+1} & = & 1 \\ x_1 & \geqslant & 0 \\ & \vdots & \\ x_{n+1} & \geqslant & 0. \end{array}$$

It is the polytope in the n-dimensional affine subspace A with the equation $x_1+\cdots+x_{n+1}=1$ bounded by the coordinate hyperplanes $x_i=0$, $i=1,\ldots,n+1$ (Figure 1.5). Prove that these hyperplanes cut A into $2^{n+1}-1$ chambers.

Hint: For a point $x=(x_1,\ldots,x_{n+1})$ in A which does not belong to any of the hyperplanes $x_i=0$, look at all possible combinations of the signs + and − of the coordinates x_i of x $i=1,\ldots,n+1$.

1.4 Isometries of $\mathbb{A}R^n$

Now let us look at the structure of $\mathbb{A}R^n$ as a metric space with the distance $r(a,b)=|\vec{ab}|$. An *isometry* of $\mathbb{A}R^n$ is a map s from $\mathbb{A}R^n$ onto $\mathbb{A}R^n$ which preserves the distance,

$$r(sa,sb)=r(a,b) \text{ for all } a,b\in\mathbb{A}R^n.$$

We denote the group of all isometries of $\mathbb{A}R^n$ by $\operatorname{Isom}\mathbb{A}R^n$.

1.4.1 Fixed points of groups of isometries

The following simple result will be used later in the case of finite groups of isometries.

Theorem 1.4.1 *Let $W<\operatorname{Isom}\mathbb{A}R^n$ be a group of isometries of $\mathbb{A}R^n$. Assume that, for some point $e\in\mathbb{A}R^n$, the orbit*

$$W\cdot e=\{\, we \mid w\in W\,\}$$

is finite. Then W fixes a point in $\mathbb{A}R^n$.

In the triangle abc the segment cd is shorter than at least one of the sides ac or bc.

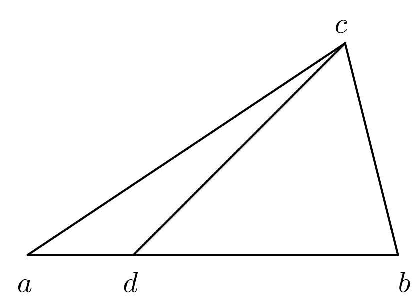


Figure 1.6: For the proof of Theorem 1.4.1

Proof[3]**.** We shall use a very elementary property of triangles stated in Figure 1.6; its proof is left to the reader.

[3]This proof is a modification of a fixed point theorem for a group acting on a space with a hyperbolic metric. J. Tits in one of his talks has attributed the proof to J. P. Serre.

Denote $E = W \cdot e$. For any point $x \in \mathbb{A}R^n$ set

$$m(x) = \max_{f \in E} r(x, f).$$

Take the point a where $m(x)$ reaches its minimum[4]. I claim that the point a is unique.

PROOF OF THE CLAIM. Indeed, if $b \neq a$ is another minimal point, take an inner point d of the segment $[a, b]$ and after that a point c such that $r(d, c) = m(d)$. We see from Figure 1.6 that, for one of the points a and b, say a,

$$m(d) = r(d, c) < r(a, c) \leqslant m(a),$$

which contradicts to the minimal choice of a.

So we can return to the proof of the theorem. Since the group W permutes the points in E and preserves the distances in $\mathbb{A}R^n$, it preserves the function $m(x)$, i.e. $m(wx) = m(x)$ for all $w \in W$ and $x \in \mathbb{A}R^n$, and thus W should fix a (unique) point where the function $m(x)$ attains its minimum. □

1.4.2 Structure of $\operatorname{Isom} \mathbb{A}R^n$

Translations. For every vector $\alpha \in \mathbb{R}^n$ one can define the map

$$\begin{array}{rcl} t_\alpha : \mathbb{A}R^n & \longrightarrow & \mathbb{A}R^n, \\ a & \mapsto & a + \alpha. \end{array}$$

The map t_α is an isometry of $\mathbb{A}R^n$; it is called the *translation through the vector* α. Translations of $\mathbb{A}R^n$ form a commutative group which we shall denote by the same symbol $\mathbb{R}^n$ as the corresponding vector space.

Orthogonal transformations. When we fix an orthonormal coordinate system in $\mathbb{A}R^n$ with the origin o, a point $a \in \mathbb{A}R^n$ can be identified with its *position vector* $\alpha = \vec{oa}$. This allows us to identify $\mathbb{A}R^n$ and $\mathbb{R}^n$. Every orthogonal linear transformation w of the Euclidean vector space $\mathbb{R}^n$, can

[4]The *existence* of the minimum is intuitively clear; an accurate proof consists of the following two observations. Firstly, the function $m(x)$, being the supremum of finite number of continuous functions $r(x, f)$, is itself continuous. Secondly, we can search for the minimum not all over the space $\mathbb{A}R^n$, but only over the set

$$\{\, x \mid r(x, f) \leqslant m(a) \text{ for all } f \in E \,\},$$

for some $a \in \mathbb{A}R^n$. This set is closed and bounded, hence compact. But a continuous function on a compact set reaches its extreme values.

be treated as a transformation of the affine space $\mathbb{A}R^n$. Moreover, this transformation is an isometry because, by the definition of an orthogonal transformation w, $(w\alpha, w\alpha) = (\alpha, \alpha)$, hence $|w\alpha| = |\alpha|$ for all $\alpha \in \mathbb{R}^n$. Therefore we have, for $\alpha = \vec{oa}$ and $\beta = \vec{ob}$,

$$r(wa, wb) = |w\beta - w\alpha| = |w(\beta - \alpha)| = |\beta - \alpha| = r(a, b).$$

The group of all orthogonal linear transformations of $\mathbb{R}^n$ is called the *orthogonal group* and denoted $\mathbb{O}_n$.

Theorem 1.4.2 *The group of all isometries of $\mathbb{A}R^n$ which fix the point o coincides with the orthogonal group $\mathbb{O}_n$.*

Proof. Let s be an isometry of $\mathbb{A}R^n$ which fixes the origin o. We have to prove that, when we treat w as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, the following conditions are satisfied: for all $\alpha, \beta \in \mathbb{R}^n$,

- $s(k\alpha) = k \cdot s\alpha$ for any constant $k \in \mathbb{R}$;
- $s(\alpha + \beta) = s\alpha + s\beta$;
- $(s\alpha, s\beta) = (\alpha, \beta)$.

If a and b are two points in $\mathbb{A}R^n$ then, by Exercise 1.4.3, the segment $[a, b]$ can be characterised as the set of all points x such that

$$r(a, b) = r(a, x) + r(x, b).$$

So the terminal point a' of the vector $c\alpha$ for $k > 1$ is the only point satisfying the conditions

$$r(o, a') = k \cdot r(0, a) \quad \text{and} \quad r(o, a) + r(a, a') = r(o, a').$$

If now $sa = b$ then, since the isometry s preserves the distances and fixes the origin o, the point $b' = sa'$ is the only point in $\mathbb{A}R^n$ satisfying

$$r(o, b') = k \cdot r(0, b) \quad \text{and} \quad r(o, b) + r(b, b') = r(o, b').$$

Hence $s \cdot k\alpha = \vec{ob'} = k\beta = k \cdot s\alpha$ for $k > 0$. The cases $k \leqslant 0$ and $0 < k \leqslant 1$ require only minor adjustments in the above proof and are left to the reader as an execise. Thus s preserves multiplication by scalars.

The additivity of s, i.e. the property $s(\alpha + \beta) = s\alpha + s\beta$, follows, in an analogous way, from the observation that the vector $\delta = \alpha + \beta$ can be constructed in two steps: starting with the terminal points a and b of the vectors α and β, we first find the midpoint of the segment $[a, b]$ as the unique point c such that

$$r(a, c) = r(c, b) = \frac{1}{2} r(a, b),$$

and then set $\delta = 2\vec{oc}$. A detailed justification of this construction is left to the reader as an exercise.

Since s preserves distances, it preserves lengths of the vectors. But from $|s\alpha| = |\alpha|$ it follows that

$$(s\alpha, s\alpha) = (\alpha, \alpha)$$

for all $\alpha \in \mathbb{R}^n$. Now we apply the additivity of s and observe that

$$\begin{aligned} ((\alpha + \beta), (\alpha + \beta)) &= (s(\alpha + \beta), s(\alpha + \beta)) \\ &= ((s\alpha + s\beta), (s\alpha + s\beta)) \\ &= (s\alpha, s\alpha) + 2(s\alpha, s\beta) + (s\beta, s\beta) \\ &= (\alpha, \alpha) + 2(s\alpha, s\beta) + (\beta, \beta). \end{aligned}$$

On the other hand,

$$((\alpha + \beta), (\alpha + \beta)) = (\alpha, \alpha) + 2(\alpha, \beta) + (\beta, \beta).$$

Comparing these two equations, we see that

$$2(s\alpha, s\beta) = 2(\alpha, \beta)$$

and

$$(s\alpha, s\beta) = (\alpha, \beta).$$

□

Theorem 1.4.3 *Every isometry of a real affine Euclidean space $\mathbb{A}R^n$ is a composition of a translation and an orthogonal transformation. The group* $\operatorname{Isom} \mathbb{A}R^n$ *of all isometries of $\mathbb{A}R^n$ is a semidirect product of the group $\mathbb{R}^n$ of all translations and the orthogonal group $\mathbb{O}_n$,*

$$\operatorname{Isom} \mathbb{A}R^n = \mathbb{R}^n \rtimes \mathbb{O}_n.$$

Proof is an almost immediate corollary of the previous result. Indeed, to comply with the definition of a semidirect product, we need to check that

$$\operatorname{Isom} \mathbb{A}R^n = \mathbb{R}^n \cdot \mathbb{O}_n, \quad \mathbb{R}^n \lhd \operatorname{Isom} \mathbb{A}R^n, \quad \text{and} \quad \mathbb{R}^n \cap \mathbb{O}_n = 1.$$

If $w \in \operatorname{Isom} \mathbb{A}R^n$ is an arbitrary isometry, take the translation $t = t_\alpha$ through the position vector $\alpha = \vec{o, wo}$ of the point wo. Then $to = wo$ and $o = t^{-1}wo$. Thus the map $s = t^{-1}w$ is an isometry of $\mathbb{A}R^n$ which fixes the origin o and, by Theorem 1.4.2, belongs to $\mathbb{O}_n$. Hence $w = ts$ and $\operatorname{Isom} \mathbb{A}R^n = \mathbb{R}^n \mathbb{O}_n$. Obviously $\mathbb{R}^n \cap \mathbb{O}_n = 1$ and we need to check only that $\mathbb{R}^n \lhd \operatorname{Isom} \mathbb{A}R^n$. But this follows from the observation that, for any orthogonal transformation s,

$$st_\alpha s^{-1} = t_{s\alpha},$$

(Exercise 1.4.5) and, consequently we have, for any isometry $w = ts$ with $t \in \mathbb{R}^n$ and $s \in \mathbb{O}_n$,

$$wt_\alpha w^{-1} = ts \cdot t_\alpha \cdot s^{-1}t^{-1} = t \cdot t_{s\alpha} \cdot t^{-1} = t_{s\alpha} \in \mathbb{R}^n.$$

□

Elations. A map $f : \mathbb{A}R^b \longrightarrow \mathbb{A}R^n$ is called an *elation* if there is a constant k such that, for all $a, b \in \mathbb{A}R^n$,

$$r(f(a), f(b)) = kr(a, b).$$

An isometry is a partial case $k = 1$ of elation. The constant k is called the *coefficient* of the elation f.

Corollary 1.4.4 *An elation of $\mathbb{A}R^n$ with the coefficient k is the composition of a translation, an orthogonal transformation and a map of the form*

$$\begin{array}{ccc} \mathbb{R}^n & \longrightarrow & \mathbb{R}^n \\ \alpha & \mapsto & k\alpha. \end{array}$$

Proof is an immediate consequence of Theorem 1.4.3. □

Exercises

1.4.1 Prove the property of triangles in $\mathbb{A}R^2$ stated in Figure 1.6.

1.4.2 BARYCENTRE. There is a more traditional approach to Theorem 1.4.1. If $F = \{ f_1, \ldots, f_k \}$ is a finite set of points in $\mathbb{A}R^n$, its *barycentre* b is a point defined by the condition

$$\sum_{j=1}^{k} b\vec{f}_j = 0.$$

1. Prove that a finite set F has a unique barycentre.
2. Further prove that the barycentre b is the point where the function

$$M(x) = \sum_{j=1}^{k} r(x, f_j)^2$$

   takes its minimal value. In particular, if the set F is invariant under the action of a group W of isometries, then W fixes the barycentre b.

Hint: Introduce orthonormal coordinates $x_1, \ldots, x_n$ and show that the system of equations

$$\frac{\partial M(x)}{\partial x_i} = 0, \quad i = 1, \ldots, n,$$

is equivalent to the equation $\sum_{j=1}^{k} x\vec{f}_j = 0$, where $x = (x_1, \ldots, x_k)$.

1.4.3 If a and b are two points in $\mathbb{A}R^n$ then the segment $[a, b]$ can be characterised as the set of all points x such that $r(a, b) = r(a, x) + r(x, b)$.

1.4.4 Draw a diagram illustrating the construction of $\alpha + \beta$ in the proof of Theorem 1.4.2, and fill in the details of the proof.

1.4.5 Prove that if t_α is a translation through the vector α and s is an orthogonal transformation then

$$st_\alpha s^{-1} = t_{s\alpha}.$$

1.4.6 Prove the following generalisation of Theorem 1.4.1: if a group $W < \operatorname{Isom} \mathbb{A}R^n$ has a bounded orbit on $\mathbb{A}R^n$ then W fixes a point.

ELATIONS.

1.4.7 Prove that an elation of $\mathbb{A}R^n$ preserves angles: if it sends points a, b, c to the points a', b', c', correspondingly, then $\angle abc = \angle a'b'c'$.

1.4.8 The group of all elations of $\mathbb{A}R^n$ is isomorphic to $\mathbb{R}^n \rtimes (\mathbb{O}_n \times \mathbb{R}^{>0})$ where $\mathbb{R}^{>0}$ is the group of positive real numbers with respect to multiplication.

1.4.9 GROUPS OF SYMMETRIES. If $\Delta \subset \mathbb{A}R^n$, the *group of symmetries* $\operatorname{Sym} \Delta$ of the set Δ consists of all isometries of $\mathbb{A}R^n$ which map Δ onto Δ. Give examples of polytopes Δ in $\mathbb{A}R^3$ such that

1. $\mathrm{Sym}\,\Delta$ acts transitively on the set of vertices of Δ but is intransitive on the set of faces;

2. $\mathrm{Sym}\,\Delta$ acts transitively on the set of faces of Δ but is intransitive on the set of vertices;

3. $\mathrm{Sym}\,\Delta$ is transitive on the set of edges of Δ but is intransitive on the set of faces.

1.5 Simplicial cones

1.5.1 Convex sets

Recall that a subset $X \subseteq \mathbb{A}R^n$ is *convex* if it contains, with any points $x, y \in X$, the segment $[x, y]$ (Figure 1.7).

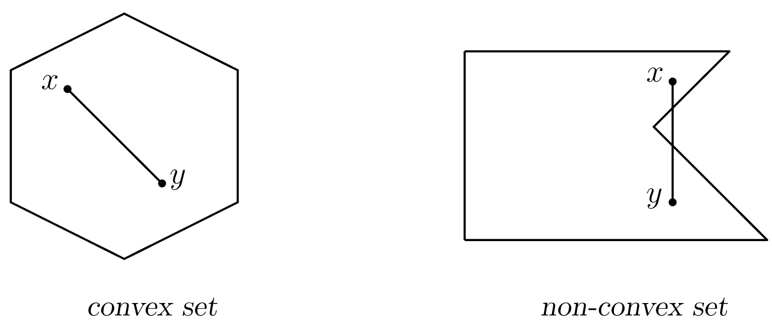


Figure 1.7: Convex and non-convex sets.

Obviously the intersection of a collection of convex sets is convex. Every convex set is connected. Affine subspaces (in particular, hyperplanes) and half spaces (open and closed) in $\mathbb{A}R^n$ are convex. If a set X is convex then so are its closure $\overline{X}$ and interior X°. If $Y \subseteq \mathbb{A}R^n$ is a subset, it *convex hull* is defined as the intersection of all convex sets containing it; it is the smallest convex set containing Y.

1.5.2 Finitely generated cones

Cones. A *cone* in $\mathbb{R}^n$ is a subset Γ closed under addition and positive scalar multiplication, that is, $\alpha + \beta \in \Gamma$ and $k\alpha \in \Gamma$ for any $\alpha, \beta \in \Gamma$ and a scalar $k > 0$. Linear subspaces and half spaces of $\mathbb{R}^n$ are cones. Every cone is convex, since it contains, with any two points α and β, the segment

$$[\alpha, \beta] = \{\, (1-t)\alpha + t\beta \mid 0 \leqslant t \leqslant 1 \,\}.$$

A cone does not necessary contains the zero vector 0; this is the case, for example, for the positive quadrant Γ in $\mathbb{R}^2$,

$$\Gamma = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; x > 0, y > 0 \right\}.$$

However, we can always add to a cone the origin 0 of $\mathbb{R}^n$: if Γ is a cone then so is $\Gamma \cup \{0\}$. It can be shown that if Γ is a cone then so is its topological closure $\overline{\Gamma}$ and interior Γ°. The intersection of a collection of cones is either a cone or the empty set.

The cone Γ *spanned* or *generated* by a set of vectors Π is the set of all non-negative linear combinations of vectors from Π,

$$\Gamma = \{ a_1\alpha_1 + \cdots + a_m\alpha_m \mid m \in \mathbb{N},\ \alpha_i \in \Pi,\ a_i \geqslant 0 \}.$$

Notice that the zero vector 0 belongs to Γ. If the set Π is finite then the cone Γ is called *finitely generated* and the set Π is a *system of generators* for Γ. A cone is *polyhedral* if it is the intersection of a finite number of closed half spaces.

The following important result can be found in most books on Linear Programming. In this book we shall prove only a very restricted special case, Proposition 1.5.6 below.

Theorem 1.5.1 *A cone is finitely generated if and only if it is polyhedral.*

Extreme vectors and edges. We shall call a set of vectors Π *positive* if, for some linear functional $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $f(\rho) > 0$ for all $\rho \in \Pi \smallsetminus \{0\}$. This is equivalent to saying that the set $\Pi \smallsetminus \{0\}$ of non-zero vectors in Π is contained in an open half space. The following property of positive sets of vectors is fairly obvious.

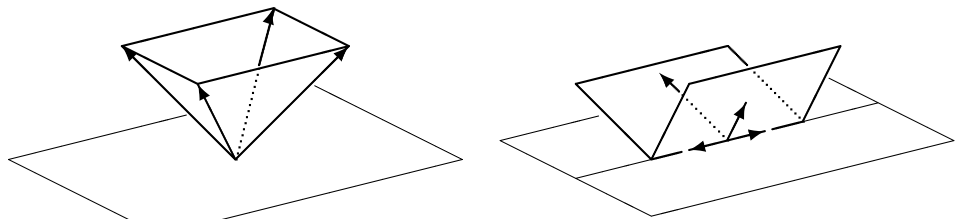
Lemma 1.5.2 *If $\alpha_1, \ldots, \alpha_m$ are non-zero vectors in a positive set Π and*

$$a_1\alpha_1 + \cdots + a_m\alpha_m = 0, \quad \textit{where all} \ \ a_i \geqslant 0,$$

then $a_i = 0$ for all $i = 1, \ldots, m$.

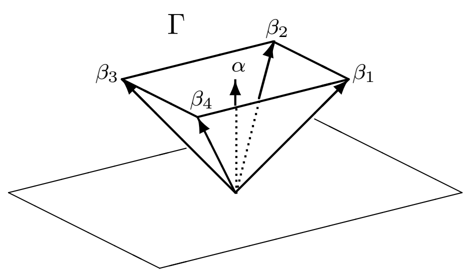
Positive cones are usually called *pointed* cones (Figure 1.8).

Let Γ be a cone in $\mathbb{R}^n$. We shall say that a vector $\epsilon \in \Gamma$ is *extreme* or *simple* in Γ if it cannot be represented as a positive linear combination which involves vectors in Γ non-collinear to ϵ, i.e. if it follows from $\epsilon = c_1\gamma_1 + \cdots + c_m\gamma_m$ where $\gamma_i \in \Gamma$ and $c_i > 0$ that $m = 1$ and $\epsilon = c_1\gamma_1$. Notice that it immediately follows from the definition that if ϵ is an extreme vector



a pointed finitely generated cone *a non-pointed finitely generated cone*

Figure 1.8: Pointed and non-pointed cones



α is a non-extreme vector in the cone Γ generated by extreme (or simple) vectors $\beta_1, \beta_2, \beta_3, \beta_4$ directed along the edges of Γ.

Figure 1.9: Extreme and non-extreme vectors.

and Π a system of generators in Γ then Π contains a vector $k\epsilon$ collinear to ϵ.

Extreme vectors in a polyhedral cone $\Gamma \subset \mathbb{R}^2$ or $\mathbb{R}^3$ have the most natural geometric interpretation: these are vectors directed along the edges of Γ. We prefer to take this property for the definition of an edge: if ϵ is an extreme vector in a polyhedral cone Γ then the cone $\Gamma \cap \mathbb{R}\epsilon$ is called an *edge* of Γ, see Figure 1.9.

1.5.3 Simple systems of generators

A finite system Π of generators in a cone Γ is said to be *simple* if it consists of simple vectors and no two distinct vectors in Π are collinear. It follows from the definition of an extreme vector that any two simple systems Π and Π' in Γ contain equal number of vectors; moreover, every vector in Π is collinear to some vector in Π', and vice versa.

Proposition 1.5.3 *Let Π be finite positive set of vectors and Γ the cone* *it generates. Assume also that Π contains no collinear vectors, that is, $\alpha = k\beta$ for distinct vectors $\alpha, \beta \in \Pi$ and $k \in \mathbb{R}$ implies $k = 0$. Then Π contains a (unique) simple system of generators.*

In geometric terms this means that a finitely generated pointed cone has finitely many edges and is generated by a system of vectors directed along the edges, one vector from each edge.

Proof. We shall prove the following claim which makes the statement of the lemma obvious.

> *A non-extreme vector can be removed from any generating set for a pointed cone Γ. In more precise terms, if the vectors $\alpha, \beta_1, \ldots, \beta_k$ of Π generate Γ and α is not an extreme vector then the vectors $\beta_1, \ldots, \beta_k$ still generate Γ.*

Proof of the claim. Let

$$\Pi = \{\, \alpha, \beta_1, \ldots, \beta_k, \gamma_1, \ldots, \gamma_l \,\},$$

where no γ_j is collinear with α. Since α is not an extreme vector,

$$\alpha = \sum_{i=1}^{k} b_i \beta_i + \sum_{j=1}^{l} c_j \gamma_j, \quad b_i \geqslant 0, c_l \geqslant 0.$$

Also, since the vectors $\alpha, \beta_1, \ldots, \beta_k$ generate the cone Γ,

$$\gamma_j = d_j \alpha + \sum_{i=1}^{k} f_{ji} \beta_i, \quad d_j \geqslant 0, f_{ji} \geqslant 0.$$

Substituting γ_i from the latter equations into the former, we have, after a simple rearrangement,

$$\left(1 - \sum_{j=1}^{l} c_j d_j\right) \alpha = \sum_{i=1}^{k} \left(b_i + \sum_{j=1}^{l} c_j f_{ji}\right) \beta_i.$$

The vector α and the vector on the right hand side of this equation both lie in the same open half space; therefore, in view of Lemma 1.5.2,

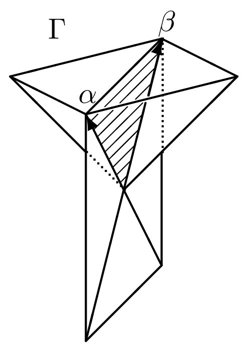
$$1 - \sum_{j=1}^{l} c_j d_j > 0$$

and

$$\alpha = \frac{1}{1 - \sum_j c_j d_j} \sum_{i=1}^{k} \left(b_i + \sum_{j=1}^{l} c_j f_{ji} \right) \beta_i$$

expresses α as a nonnegative linear combination of β_i's. Since the vectors $\alpha, \beta_1, \ldots, \beta_k$ generate Γ, the vectors $\beta_1, \ldots, \beta_k$ also generate Γ. □

The following simple lemma has even simpler geometric interpretation: the plane passing through two edges of a cone cuts in it the cone spanned by these two edges, see Figure 1.10.



The intersection of a cone Γ with the plane spanned by two simple vectors α and β is the coned generated by α and β.

Figure 1.10: For the proof of Lemma 1.5.4

Lemma 1.5.4 *Let α and β be two distinct extreme vectors in a finitely generated cone Γ. Let P be the plane (2-dimensional vector subspace) spanned by α and β. Then $\Gamma_0 = \Gamma \cap P$ is the cone in P spanned by α and β.*

Proof. Assume the contrary; let $\gamma \in \Gamma_0$ be a vector which does not belong to the cone spanned by α and β. Since α and β form a basis in the vector space P,

$$\gamma = a'\alpha + b'\beta,$$

and by our assumption one of the coefficients a' or b' is negative. We can assume without loss of generality that $b' < 0$.

Let $\alpha, \beta, \gamma_1, \ldots, \gamma_m$ be the simple system in Γ. Since $\gamma \in \Gamma$,

$$\gamma = a\alpha + b\beta + c_1\gamma_1 + \cdots + c_m\gamma_m,$$

where all the coefficients a, b, $c_1, \ldots, c_m$ are non-negative. Comparing the two expressions for γ, we have

$$(a - a')\alpha + (b - b')\beta + c_1\gamma_1 + \cdots + c_m\gamma_m = 0.$$

Notice that $b - b' > 0$; if $a - a' \geqslant 0$ then we get a contradiction with the assumption that the cone Γ is pointed. Therefore $a - a' < 0$ and

$$\alpha = \frac{1}{a' - a}\left((b - b')\beta + c_1\gamma_1 + \cdots + c_m\gamma_m\right)$$

expresses α as a non-negative linear combination of the rest of the simple system. This contradiction proves the lemma. □

1.5.4 Duality

If Γ is a cone, the *dual cone* Γ^* is the set

$$\Gamma^* = \{\, \chi \in \mathbb{R}^n \mid (\chi, \gamma) \leqslant 0 \text{ for all } \gamma \in \Gamma \,\}.$$

It immediately follows from this definition that the set Γ^* is closed with respect to addition and multiplication by positive scalars, so the name 'cone' for it is justified. Also, the dual cone Γ^*, being the interesection of closed half-spaces $(\chi, \gamma) \leqslant 0$, is closed in topological sense.

The following theorem plays an extremely important role in several branches of Mathematics: Linear Progamming, Functional Analysis, Convex Geometry. We shall not use or prove it in its full generality, proving instead a simpler partial case.

Theorem 1.5.5 (The Duality Theorem for Polyhedral Cones) *If Γ is a polyhedral cone, then so is Γ^*. Moreover, $(\Gamma^*)^* = \Gamma$.*

Recall that polyhedral cones are closed by definition.

1.5.5 Duality for simplicial cones

Simplicial cones. A finitely generated cone $\Gamma \subset \mathbb{R}^n$ is called *simplicial* if it is spanned by n linearly independent vectors $\rho_1, \ldots, \rho_n$. Denote $\Pi = \{\, \rho_1, \ldots, \rho_n \,\}$.

We shall prove the Duality Theorem 1.5.5 in the special case of simplicial cones, and obtain, in the course of the proof, very detailed information about their structure.

First of all, notice that if the cone Γ is generated by a finite set $\Pi = \{\, \rho_1, \ldots, \rho_n \,\}$ then the inequalities

$$(\chi, \gamma) \leqslant 0 \quad \text{for all } \gamma \in \Gamma$$

are equivalent to

$$(\chi, \rho_i) \leqslant 0, \quad i = 1, \ldots, n.$$

Hence the dual cone Γ^* is the intersection of the closed subspaces given by the inequalities

$$(\chi, \rho_i) \leqslant 0, \quad i = 1, \ldots, n.$$

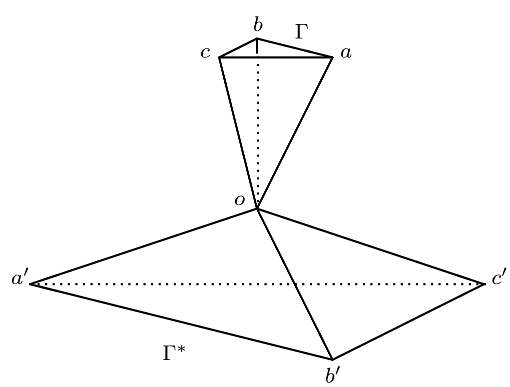
We know from Linear Algebra that the conditions

$$(\rho_i^*, \rho_j) = \begin{cases} -1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

uniquely determine n linearly independent vectors $\rho_1^*, \ldots, \rho_n^*$ (see Exercises 1.5.2 and 1.5.3). We shall say that the basis $\Pi^* = \{\, \rho_1^*, \ldots, \rho_n^* \,\}$ is *dual*[5] to the basis $\rho_1, \ldots, \rho_n$. If we write a vector $\chi \in \mathbb{R}^n$ in the basis Π^*,

$$\chi = y_1^* \rho_1^* + \cdots + y_n^* \rho_n^*,$$

then $(\chi, \rho_i) = -y_i^*$ and $\chi \in \Gamma^*$ if and only if $y_i \geqslant 0$ for all i, which means that $\chi \in \Gamma^*$. So we proved the following partial case of the Duality Theorem, illustrated by Figure 1.11.



The simplicial cones Γ *and* Γ^* *are dual to each other:*

$oa \perp b'oc'$, $ob \perp c'oa'$, $oc \perp a'ob'$,

$oa' \perp boc$, $ob' \perp coa$, $oc' \perp aob$.

Figure 1.11: Dual simplicial cones.

Proposition 1.5.6 *If* Γ *is the simplicial cone spanned by a basis* Π *of* $\mathbb{R}^n$ *then the dual cone* Γ^* *is also simplicial and spanned by the dual basis* Π^**. Applying this property to* Γ^* *we see that* $\Gamma = (\Gamma^*)^*$ *is the dual cone to* Γ^* *and coincides with the intersection of the closed half spaces*

$$(\chi, \rho_i^*) \leqslant 0, \quad i = 1, \ldots, n.$$

[5]We move a little bit away from the traditional terminology, since the dual basis is usually defined by the conditions

$$(\rho_i^*, \rho_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

1.5.6 Faces of a simplicial cone

Denote by H_i the hyperplane $(\chi, \rho_i^*) = 0$. Notice that the cone Γ lies in one closed half space determined by H_i. The intersection $\Gamma_k = \Gamma \cap H_k$ consists of all vectors of the form $\chi = y_1\rho_1 + \cdots + y_n\rho_n$ with non-negative coordinates y_i, $i = 1, \ldots, n$, and zero k-th coordinate, $y_k = 0$. Therefore Γ_k is the simplicial cone in the $n-1$-dimensional vector space $(\chi, \rho_k^*) = 0$ spanned by the vectors ρ_i, $i \neq k$. The cones Γ_k are called *facets* or $(n-1)$-dimensional faces of Γ.

More generally, if we denote $I = \{1, \ldots, n\}$ and take a subset $J \subset I$ of cardinality m, then the $(n-m)$-*dimensional face* Γ_J of Γ can be defined in two equivalent ways:

- Γ_J is the cone spanned by the vectors ρ_i, $i \in I \smallsetminus J$.
- $\Gamma_J = \Gamma \cap \bigcap_{j\in J} H_j$.

It follows from their definition that edges are 1-dimensional faces.

If we define the faces Γ_J^* in an analogous way then we have the formula

$$\Gamma_J^* = \{\, \chi \in \Gamma^* \mid (\chi, \gamma) = 0 \text{ for all } \gamma \in \Gamma_{I\smallsetminus J} \,\}.$$

Abusing terminology, we shall say that the face Γ_J^* of Γ^* is *dual* to the face $\Gamma_{I\smallsetminus J}$ of Γ. This defines a one-to-one correspondence between the faces of the simplicial cone Γ and its dual Γ^*.

In particular, the edges of Γ are dual to facets of Γ^*, and the facets of Γ are dual to edges of Γ^*.

We shall use also the Duality Theorem for cones Γ spanned by $m < n$ linearly independent vectors in $\mathbb{R}^n$. The description of Γ^* in this case is an easy generalisation of Proposition 1.5.6; see Exercise 1.5.4.

Exercises

1.5.1 Let X be an arbitrary positive set of vectors in $\mathbb{R}^n$. Prove that the set

$$X^* = \{\, \alpha \in \mathbb{R}^n \mid (\alpha, \gamma) \leqslant 0 \,\}$$

is a cone. Show next that X^* contains a non-zero vector and that X is contained in the cone $(X^*)^*$.

1.5.2 Dual basis. Let $\epsilon_1, \ldots, \epsilon_n$ be an orthonormal basis and $\rho_1, \ldots, \rho_n$ a basis in $\mathbb{R}^n$. Form the matrix $R = (r_{ij})$ by the rule $r_{ij} = (\rho_i, \epsilon_j)$, so that $\rho_i = \sum_{j=1}^{n} r_{ij}\epsilon_j$. Notice that R is a non-degenerate matrix. Let $\rho = y_1\epsilon_1 + \cdots + y_n\epsilon_n$. For each value of i, express the system of simultaneous equations

$$(\rho, \rho_j) = \begin{cases} -1 & \text{if } i = j \\ \;\;\,0 & \text{if } i \neq j \end{cases}$$

in matrix form and prove that it has a unique solution. This will prove the existence of the basis dual to $\rho_1, \dots, \rho_n$.

1.5.3 A FORMULA FOR THE DUAL BASIS. In notation of Exercise 1.5.2, prove that the dual basis $\{ \rho_i^* \}$ can be determined from the formula

$$\rho_j^* = -\frac{1}{\det R} \begin{vmatrix} r_{11} & \dots & r_{1,j-1} & \epsilon_1 & r_{1,j+1} & \dots & r_{1,n} \\ r_{21} & \dots & r_{2,j-1} & \epsilon_2 & r_{2,j+1} & \dots & r_{2,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ r_{i,1} & \dots & r_{i,j-1} & \epsilon_i & r_{i,j+1} & \dots & r_{i,n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ r_{n,1} & \dots & r_{n,j-1} & \epsilon_n & r_{n,j+1} & \dots & r_{n,n} \end{vmatrix}.$$

Notice that in the case $n = 3$ we come to the formula

$$\rho_1^* = -\frac{1}{(\rho_1, \rho_2, \rho_3)} \rho_2 \times \rho_3, \quad \rho_2^* = -\frac{1}{(\rho_1, \rho_2, \rho_3)} \rho_3 \times \rho_1, \quad \rho_3^* = -\frac{1}{(\rho_1, \rho_2, \rho_3)} \rho_1 \times \rho_2,$$

where $(\ , \ , \ )$ denotes the scalar triple product and $\times$ the cross (or vector) product of vectors.

1.5.4 Let Γ be a cone in $\mathbb{R}^n$ spanned by a set Π of m linearly independent vectors $\rho_1, \dots, \rho_m$, with $m < n$. Let U be the vector subspace spanned by Π.

Then Γ is a simplicial cone in U; we denote its dual in U as Γ', and set Γ^* to be the dual cone for Γ in V. Let also $\Pi' = \{ \rho_1', \dots, \rho_m' \}$ be the basis in U dual to the basis Π. We shall use in the sequel the following properties of the cone Γ^*.

1. For any set $A \in \mathbb{R}^n$, define
$$A^\perp = \{ \chi \in \mathbb{R}^n \mid (\chi, \alpha) = 0 \text{ for all } \gamma \in A \}.$$
Check that $A^\perp$ is a linear subspace of $\mathbb{R}^n$. Prove that $\dim \Gamma^\perp = n - m$.
Hint: $\Gamma^\perp = U^\perp$.

2. Γ^* is the intersection of the closed half spaces defined by the inequalities $(\chi, \rho_i) \leqslant 0$, $i = 1, \dots, m$.

3. $\Gamma^* = \Gamma' + \Gamma^\perp$; this set is, by definition,
$$\Gamma' + \Gamma^\perp = \{ \kappa + \chi \mid \kappa \in \Gamma', \, \chi \in \Gamma^\perp \}.$$

4. $(\Gamma^*)^* = \Gamma$.

5. Let H_i and H_i^* be the hyperplanes in V given by the equations $(\chi, \rho_i') = 0$ and $(\chi, \rho_i) = 0$, correspondingly. Denote $I = \{ 1, \dots, m \}$ and set, for $J \subseteq I$,
$$\Gamma_J = \Gamma \cap \bigcap_{j \in J} H_j, \ \ \Gamma_J^* = \Gamma^* \cap \bigcap_{j \in J} H_j^* \ \text{ and } \ \Gamma_J' = \Gamma' \cap \bigcap_{j \in J} H_j^*.$$
Prove that $\Gamma_J^* = \Gamma_J' + \Gamma^\perp$.

6. The cones Γ_J and Γ_J^* are called *faces* of the cones Γ and Γ^*, correspondingly. There is a one-to-one correspondence between the set of k-dimensional faces of Γ, $k = 1, \dots, m-1$, and $n-k$ dimensional faces of Γ^*, defined by the rule $\Gamma_J^* = \Gamma \cap \Gamma_{I \smallsetminus J}^{\perp}$. If we treat Γ as its own m-dimensional face $\Gamma_\emptyset$, then it corresponds to $\Gamma_I^* = \Gamma^\perp$.

Chapter 2

Mirrors, Reflections, Roots

2.1 Mirrors and reflections

Mirrors and reflections. Recall that a *reflection* in an affine real Euclidean space $\mathbb{A}R^n$ is a nonidentity isometry s which fixes all points of some affine hyperplane (i.e. affine subspace of codimension 1) H of $\mathbb{A}R^n$. The hyperplane H is called the *mirror* of the reflection s and denoted H_s. Conversely, the reflection s will be sometimes denoted as $s = s_H$.

Lemma 2.1.1 *If s is a reflection with the mirror H then, for any point $\alpha \in \mathbb{A}R^n$,*

- *the segment $[s\alpha, \alpha]$ is normal to H and H intersects the segment in its midpoint;*
- *H is the set of points fixed by s;*
- *s is an involutary transformation*[1]*, that is, $s^2 = 1$.*

In particular, the reflection s is uniquily determined by its mirror H, and vice versa.

Proof. Choose some point of H for the origin o of a orthonormal coordinate system, and identify the affine space $\mathbb{A}R^n$ with the underlying real Euclidean vector space $\mathbb{R}^n$. Then, by Theorem 1.4.2, s can be identified with an orthogonal transformation of $\mathbb{R}^n$. Since s fixes all points in H, it has at least $n-1$ eigenvalues 1, and, since s is non identity, the only possibility for the remaining eigenvalue is -1. In particular, s is diagonalisable and has order 2, that is $s^2 = 1$ and $s \neq 1$. It also follows from here that H is the set of all point fixed by s.

[1] A non-identity element g of a group G is called an *involution* if it has order 2. Hence s is an involution.

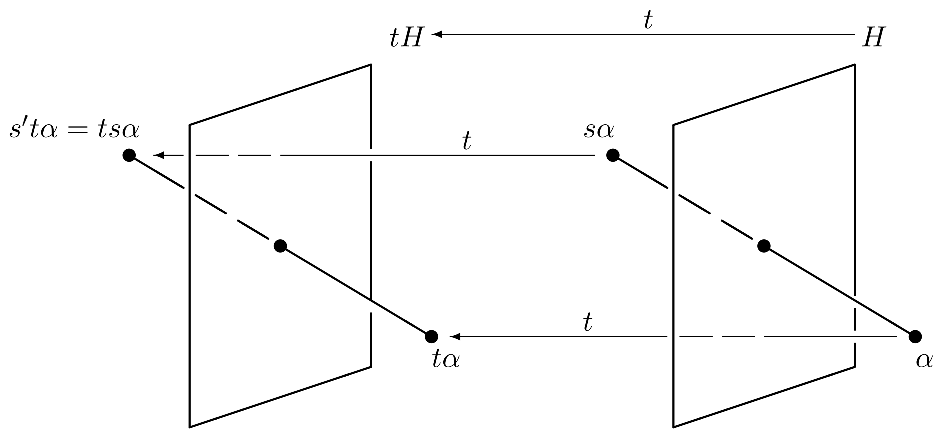


Figure 2.1: *For the proof of Lemma* 2.1.3*: If* s *is the reflection in the mirror* H *and* t *is an isometry then the reflection* s' *in the mirror* tH *can be found from the condition* $s't = ts$ *hence* $s' = tst^{-1}$.

If now we consider the vector $s\alpha - \alpha$ directed along the segment $[s\alpha, \alpha]$ then

$$s(s\alpha - \alpha) = s^2\alpha - s\alpha = \alpha - s\alpha,$$

which means that the vector $s\alpha - \alpha$ is an eigenvector of s for the eigenvalue -1. Hence the segment $[s\alpha, \alpha]$ is normal to H. Its midpoint $\frac{1}{2}(s\alpha + \alpha)$ is s-invariant, since

$$s\frac{1}{2}(s\alpha + \alpha) = \frac{1}{2}(s^2\alpha + s\alpha) = \frac{1}{2}(s\alpha + \alpha),$$

hence belongs to H. □

In the course of the proof of the previous lemma we have also shown

Lemma 2.1.2 *Reflections in* $\mathbb{A}R^n$ *which fix the origin* o *are exactly the orthogonal transformations of* $\mathbb{R}^n$ *with* $n-1$ *eigenvalues* 1 *and one eigenvalue* -1*; their mirrors are eigenspaces for the eigenvalue* 1.

We say that the points $s\alpha$ and α are *symmetric* in H. If $X \subset A$ then the set sX is called the *reflection* or the *mirror image* of the set X in the mirror H.

Lemma 2.1.3 *If* t *is an isometry of* $\mathbb{A}R^n$, s *the reflection in the mirror* H *and* s' *is the reflection in* tH *then* $s' = tst^{-1}$.

Proof. See Figure 2.1. Alternatively, we may argue as follows.

We need only show that tst^{-1} is a non-identity isometry which fixes tH. Since tst^{-1} is a composition of isometries, it is clearly an isometry. If $\alpha \in tH$, then $t^{-1}\alpha \in H$, hence s fixes $t^{-1}\alpha$, hence $tst^{-1}\alpha = \alpha$. If $\alpha \notin tH$, then $t^{-1}\alpha \notin H$, hence s does not fix $t^{-1}\alpha$, hence $tst^{-1}\alpha \neq \alpha$. □

Exercises

Reflections and rotations in $\mathbb{R}^2$.

2.1.1 Prove that every 2 orthogonal matrix A over $\mathbb{R}$ can be written in one of the forms

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

or

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

depending on whether A has determinant 1 or -1.

2.1.2 Prove that if, in the notation of the previous Exercise, $\det A = 1$ then A is the matrix of the rotation through the angle θ about the origin, counterclockwise.

2.1.3 Prove that if $\det A = -1$ then A is the matrix of a reflection.

2.1.4 Check that

$$u = \begin{pmatrix} \cos\phi/2 \\ \sin\phi/2 \end{pmatrix} \text{ and } v = \begin{pmatrix} -\sin\phi/2 \\ \cos\phi/2 \end{pmatrix}$$

are eigenvectors with the eigenvalues 1 and -1 for the matrix

$$\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}.$$

2.1.5 Use trigonometric identities to prove that

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \cdot \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} = \begin{pmatrix} \cos(\phi+\psi) & -\sin(\phi+\psi) \\ \sin(\phi+\psi) & \cos(\phi+\psi) \end{pmatrix}.$$

Give a geometric interpretation of this fact.

Finite groups of orthogonal transformations in 2 dimensions.

2.1.6 Prove that any finite group of rotations of the Euclidean plane $\mathbb{R}^2$ about the origin is cyclic.

Hint: It is generated by a rotation through the smallest angle.

2.1.7 Prove that if r is a rotation of $\mathbb{R}^2$ and s a reflection then sr is a reflection, in particular, $|sr| = 2$. Deduce from this the fact that s *inverts* r, i.e. $srs^{-1} = r^{-1}$.

2.1.8 If G is a finite group of orthogonal transformations of the 2-dimensional Euclidean space $\mathbb{R}^2$ then the map

$$\begin{array}{rcl} \det : G & \longrightarrow & \{\,1, -1\,\} \\ A & \mapsto & \det A \end{array}$$

is a homomorphism with the kernel R consisting of all rotations contained in G. If $R \neq G$ then $|G : R| = 2$ and all elements in $G \smallsetminus R$ are reflections.

2.1.9 Prove that the product of two reflections in $\mathbb{R}^2$ (with a common fixed point at the origin) is a rotation through twice the angle between their mirrors.

Involutary orthogonal transformations in three dimensions.

2.1.10 In $\mathbb{R}^3$ there are three, up to conjugacy of matrices, involutive orthogonal transformations, with the eigenvalues $1, 1, -1$ (reflections), $1, -1, -1$ and $-1, -1, -1$. Give a geometric interpretation of the two latter transformations.

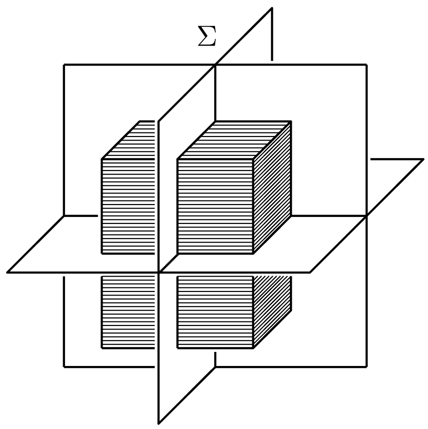
2.2 Systems of mirrors

Assume now that we are given a solid $\Delta \subset \mathbb{A}R^n$. Consider the set Σ of all mirrors of symmetry of Δ, i.e. the mirrors of reflections which send Δ to Δ. The reader can easily check (Exercise 2.2.1) that Σ is a *closed system of mirrors* in the sense of the following definition: a system of hyperplanes (mirrors) in A is called *closed* if, for any two mirrors H_1 and H_2 in Σ, the mirror image of H_2 in H_1 also belongs to Σ (see Figure 2.)

Slightly abusing language, we shall call a finite closed system Σ of mirrors simply a *system of mirrors*.

Systems of mirrors are the most natural objects. The reader most likely has seen them when looking into a kaleidoscope[2]; and, of course, everybody

[2]My special thanks are due to Dr. Robert Sandling who lent me, for demonstration to my students, a fascinating old fashioned kaleidoscope. It contained three mirrors arranged as the side faces of a triangular prism with an equilateral base and produced the mirror system of type $\tilde{A}_2$.



The system Σ of mirrors of symmetry of a geometric body Δ is closed*: the reflection of a mirror in another mirror is a mirror again. Notice that if Δ is compact then all mirrors intersect in a common point.*

Figure 2.2: A closed system of mirrors.

has seen a mirror[3]. We are interested in the study of finite closed systems of mirrors and other, closely related objects—root systems and finite groups generated by reflections.

Systems of reflections. If Σ is a system of mirrors, the set of all reflections in mirrors of Σ will be refered to as a *closed system of reflections*. In view of Lemma 2.1.3, a set S of reflections forms a closed system of reflections if and only if $s^t \in S$ for all $s, t \in S$. Here s^t is the standard, in group theory, abbreviation for conjugation: $s^t = t^{-1}st$. Recall that conjugation by any element t is an automorphism of any group containing t: $(xy)^t = x^t y^t$.

Lemma 2.2.1 *A finite closed system of reflections generates a finite group of isometries.*

Proof. This result is a partial case of the following elementary group theoretic property.

> *Let W be a group generated by a finite set S of involutions such that $s^t \in S$ for all $s, t \in S$. Then W is finite.*

Indeed, since $s \in S$ are involutions, $s^{-1} = s$. Let $w \in W$ and find the shortest expression $w = s_1 \cdots s_k$ of w as a product of elements from S. If

[3]We cannot resist temptation and recall an old puzzle: why is it that the mirror changes left and right but does not change up and down?

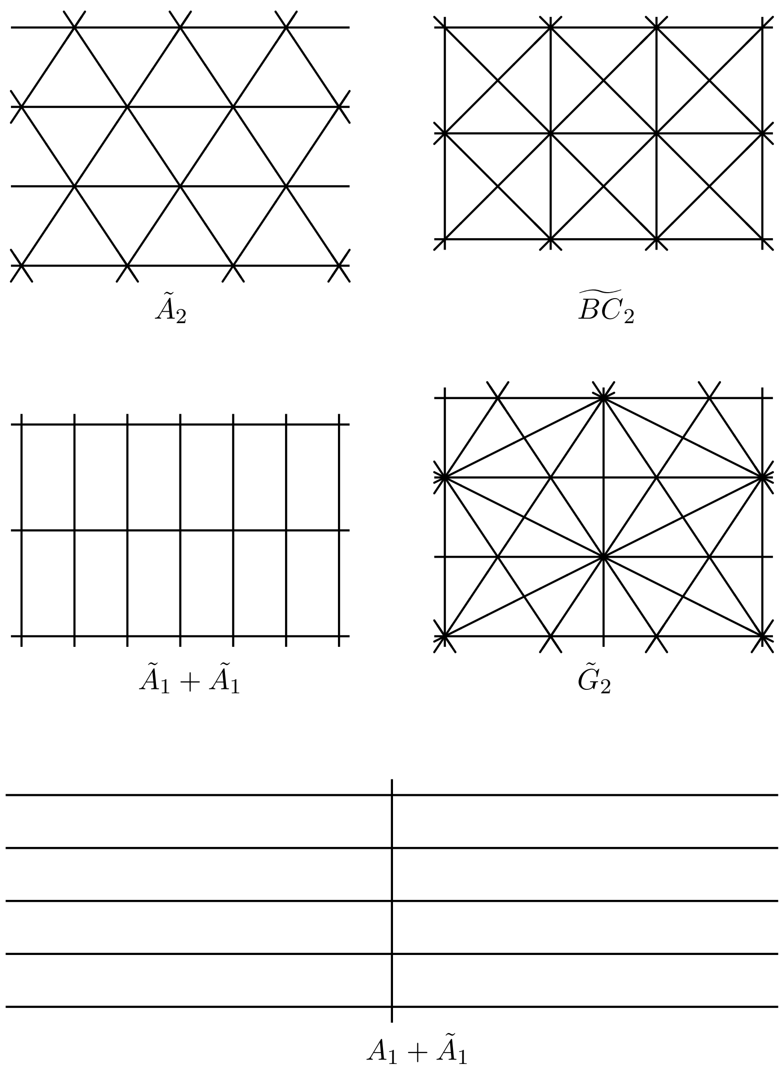


Figure 2.3: *Examples of infinite closed mirror systems in* $\mathbb{A}R^2$ *with their traditional notation: tesselations of the plane by congruent equilateral triangles* ($\tilde{A}_2$), *isosceles right triangles* ($\widetilde{BC}_2$), *rectangles* ($\tilde{A}_1 + \tilde{A}_1$), *triangles with the angles* $\pi/2$, $\pi/3$, $\pi/6$ ($\tilde{G}_2$), *infinite half stripes* ($A_1 + \tilde{A}_1$).

the word $s_1 \cdots s_k$ contains two occurrences of the same involution $s \in S$ then

$$\begin{aligned} w &= s_1 \cdots s_i s s_{i+1} \cdots s_j s s_{j+1} \cdots s_k \\ &= s_1 \cdots s_i (s_{i+1} \cdots s_j)^s s_{j+1} \cdots s_k \\ &= s_1 \cdots s_i s_{i+1}^s \cdots s_j^s s_{j+1} \cdots s_k \\ &= s_1 \cdots s_i s'_{i+1} \cdots s'_j s_{j+1} \cdots s_k, \end{aligned}$$

where all $s'_l = s_l^s$ belong to S and the resulting expression is shorter then the original. Therefore all shortest expressions of elements from W in terms of generators $s \in S$ contain no repetition of symbols. Therefore the length of any such expression is at most $|S|$, and, counting the numbers of expressions of length $0, 1, \ldots, |S|$, we find that their total number is at most

$$1 + |S| + |S|^2 + \cdots + |S|^{|S|}.$$

Hence W is finite. □

Finite reflection groups. A group-theoretic interpretation of closed systems of mirrors comes in the form of a *finite reflection group*, i.e. a finite group W of isometries of an affine Euclidean space A generated by reflections.

Let s be a reflection in W and $s^W = \{ wsw^{-1} \mid w \in W \}$ its conjugacy class. Form the set of mirrors $\Sigma = \{ H_t \mid t \in s^W \}$. Then it follows from Lemma 2.1.3 that Σ is a mirror system: if $H_r, H_t \in \Sigma$ then the reflection of H_r in H_t is the mirror H_{r^t}. Thus s^W is a closed system of reflections. The same observation is valid for any *normal* set S of reflections in W, i.e. a set S such that $s^w \in S$ for all $s \in S$ and $w \in W$. We shall show later that if the reflection group W arises from a closed system of mirrors Σ then every reflection in W is actually the reflection in one of the mirrors in Σ.

Since W is finite, all its orbits are finite and W fixes a point by virtue of Theorem 1.4.1. We can take this fixed point for the origin of an orthonormal coordinate system and, in view of Theorem 1.4.2, treat W as a group of linear orthogonal transformations.

If W is the group generated by the reflections in the finite closed system of mirrors Σ then the fixed points of W are fixed by every reflection in a mirror from Σ hence belong to each mirror in Σ. Thus we proved

Theorem 2.2.2 (1) *A finite reflection group in $\mathbb{A}R^n$ has a fixed point.*

(2) *All the mirrors in a finite closed system of mirrors in $\mathbb{A}R^n$ have a point in common.*

Since we are interested in finite closed system of mirros and finite groups generated by reflections, this result allows us to assume without loss of generality that all mirrors pass through the origin of $\mathbb{R}^n$. So we can forget about the affine space $\mathbb{A}R^n$ and work entirely in the Euclidean vector space $V = \mathbb{R}^n$.

Exercises

Systems of mirrors.

2.2.1 Prove that if Δ is a subset in $\mathbb{A}R^n$ then the system Σ of its mirrors of symmetry is closed.

Hint: If M and N are two mirrors in Σ with the reflections s and t, then, in view of Lemma 2.1.3*, the mirror image of M in N is the mirror of the reflection s^t. If s and t map Δ onto Δ then so does s^t.*

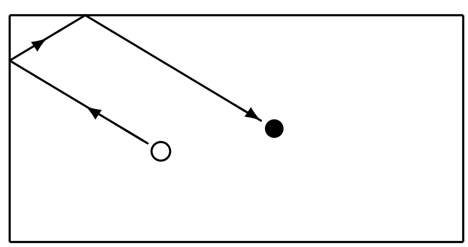


Figure 2.4: Billiard, for Exercise 2.2.2.

2.2.2 Two balls, white and black, are placed on a billiard table (Figure 2.4). The white ball must bounce off two cushions of the table and then strike the black one. Find its trajectory.

2.2.3 Prove that a ray of light reflecting from two mirrors forming a corner will eventually get out of the corner (Figure 2.5). If the angle formed by the mirrors is α, what is the maximal possible number of times the ray would bounce off the sides of the corner?

2.2.4 Prove that the angular reflector made of three pairwise perpendicular mirrors in $\mathbb{R}^3$ sends a ray of light back in the direction exactly opposite to the one it came from, (Figure 2.6).

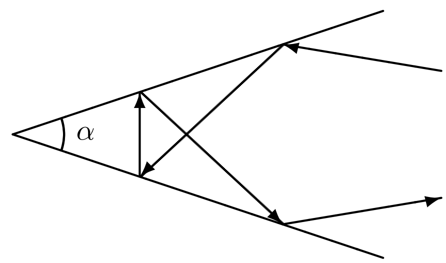


Figure 2.5: For Exercise 2.2.3.

REFLECTIONS AND LINEAR ALGEBRA.

2.2.5 We say that a subpace U of the real Euclidean space V is *perpendicular* to the subspace W and write $U \perp W$ if $U = (U \cap W) \oplus U'$ where U' is orthogonal to W, i.e. $(u, w) = 0$ for all $u \in U'$ and $w \in W$. Prove that this relation is symmetric; $U \perp W$ if and only if $W \perp U$.

2.2.6 Prove that if a reflection s leaves a subspace $U < V$ invariant then U is perpendicular to the mirror H_s of the reflection s.

2.2.7 Prove that two reflections s and t commute, that is, $st = ts$, if and only if their mirrors are perpendicular to each other.

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2.2.8 Prove that the product of two reflections in $\mathbb{A}R^2$ with parallel mirrors is a parallel translation. What is the translation vector?

2.2.9 If a bounded figure in the Euclidean plane $\mathbb{A}R^2$ has a center of symmetry and a mirror of symmetry then it has two perpendicular mirrors of symmetry. Is the same true in $\mathbb{A}R^3$?

2.3 Dihedral groups

In this section we shall study finite groups generated by two involutions.

Theorem 2.3.1 *There is a unique, up to isomorphism, group W generated by two involutions s and t such that their product st has order n. Furthermore,*

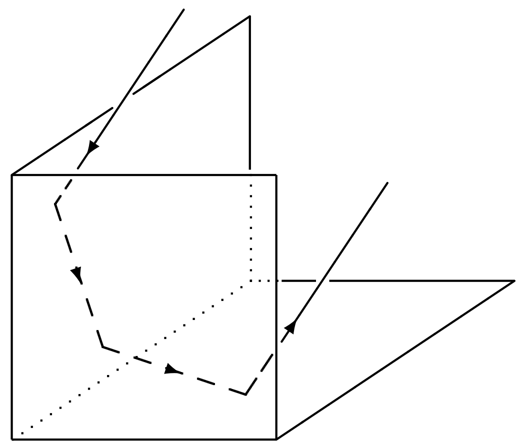


Figure 2.6: *Angular reflector* (*for Exercise* 2.2.4).

(1) *W is finite and $|W| = 2n$.*

(2) *If $r = st$ then the cyclic group $R = \langle r \rangle$ generated by r is a normal subgroup of W of index* 2.

(3) *Every element in $W \smallsetminus R$ is an involution.*

We shall denote the group W as D_{2n}, call it the *dihedral group* of order $2n$ and write

$$D_{2n} = \langle s, t \mid s^2 = t^2 = (st)^n = 1 \rangle.$$

This standard group-theoretical notation means that the group D_{2n} is generated by two elements s and t such that any identity relating them to each other is a consequence of the *generating relations*

$$s^2 = 1, \ t^2 = 1, \ (st)^n = 1.$$

The words 'consequence of the generating relations' are given precise meaning in the theory of groups given by generators and relations, a very well developed chapter of the general theory of groups. We prefer to use them in a informal way which will be always clear from context.

Proof of the Theorem 2.3.1. First of all, notice that, since $s^2 = t^2 = 1$,

$$s^{-1} = s \text{ and } t^{-1} = t.$$

In particular, $(st)^{-1} = t^{-1}s^{-1} = ts$. Set $r = st$. Then

$$r^t = trt = t \cdot st \cdot t = ts = r^{-1}$$

and analogously $r^s = r^{-1}$.

Since $s = rt$, the group W is generated by r and t and every element w in W has the form

$$w = r^{m_1}t^{k_1} \cdots r^{m_l}t^{k_l},$$

where m_i takes the values $0, 1, \ldots, n-1$ and k_i is 0 or 1. But one can check that, since $trt = r^{-1}$,

$$tr = r^{-1}t,$$

$$tr^m = r^{-m}t$$

and

$$t^k r^m = r^{(-1)^k m} t^k.$$

Hence

$$(r^{m_1}t^{k_1})(r^{m_2}t^{k_2}) = r^m t^k \tag{2.1}$$

where

$$k = k_1 + k_2, \quad m = m_1 + (-1)^{k_1} m_2.$$

Therefore every element in W can be written in the form

$$w = r^m t^k, \; m = 0, \ldots, n-1, \; k = 0, 1.$$

Furthemore, this presentation is unique. Indeed, assume that

$$r^{m_1}t^{k_1} = r^{m_2}t^{k_2}$$

where $m_1, m_2 \in \{0, \ldots, n-1\}$ and $k_1, k_2 \in \{0, 1\}$. If $k_1 = k_2$ then $r^{m_1} = r^{m_2}$ and $m_1 = m_2$. But if $k_1 \neq k_2$ then

$$r^{m_1 - m_2} = t.$$

Denote $m = m_1 - m_2$. Then $m < n$ and $r^m = (st)^m = t$. If $m = 0$ then $t = 1$, which contradicts to our assumption that $|t| = 2$. Now we can easily get a final contradiction:

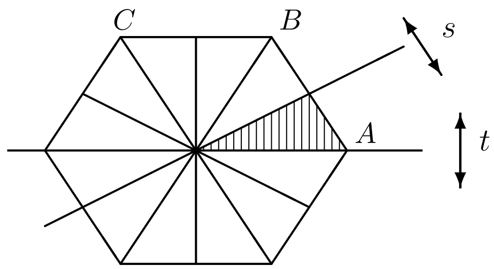
$$st \cdot st \cdots st = t$$

implies

$$s \cdot ts \cdots ts \cdots ts = 1.$$

The word on the left contains an odd number of elements s and t. Consider the element r in the very center of the word; r is either s or t. Hence the previous equation can be rewritten as

$$sts \cdots r \cdots sts = [sts \cdots]r[sts \cdots]^{-1} = 1,$$



The group of symmetries of the regular n-gon Δ is generated by two reflections s and t in the mirrors passing through the midpoint and a vertex of a side of Δ.

Figure 2.7: For the proof of Theorem 2.3.2.

which implies $r = 1$, a contradiction.

Since elements of w can be represented by expressions $r^m t^k$, and in a unique way, we conclude that $|W| = 2n$ and

$$W = \{\, r^m t^k \mid m = 0, 1, \ldots, n-1,\ k = 0, 1 \,\},$$

with the multiplication defined by Equation 2.1. This proves existence and uniqueness of D_{2n}.

Since $|r| = n$, the subgroup $R = \langle r \rangle$ has index 2 in W and hence is normal in W. If $w \in W \setminus R$ then $w = r^m t$ for some m, and a direct computation shows

$$w^2 = r^m t \cdot r^m t = r^{-m+m} t^2 = 1.$$

Since $w \neq 1$, w is an involution. □

Theorem 2.3.2 *The group of symmetries* $\operatorname{Sym}\Delta$ *of the regular n-gon* Δ *is isomorphic to the dihedral group D_{2n}.*

Proof. Denote $W = \operatorname{Sym}\Delta$. The mirrors of symmetry of the polygon Δ cut it into $2n$ triangle slices[4], see Figure 2.7. Notice that any two adjacent slices are interchanged by the reflection in their common side. Therefore W acts transitively on the set S of all slices. Also, observe that only the identity symmetry of Δ maps a slice onto itself. By the well-known formula for the length of a group orbit

$$|W| = \begin{pmatrix} \text{number} \\ \text{of slices} \end{pmatrix} \cdot \begin{pmatrix} \text{order of the} \\ \text{stabiliser} \\ \text{of a slice} \end{pmatrix} = 2n \cdot 1 = 2n.$$

[4]Later we shall use for them the more terms *fundamental regions* or *chambers*.

Next, if s and t are reflections in the side mirrors of a slice, then their product st is a rotation through the angle $2\pi/n$, which can be immediately seen from the picture: st maps[5] the vertex A to B and B to C. By Theorem 2.3.1, $|\langle s,t\rangle| = 2n$; hence $W = \langle s,t\rangle$ is the dihedral group of order $2n$. □

Exercises

2.3.1 Prove that the dihedral group D_6 is isomorphic to the symmetric group Sym_3.

2.3.2 The centre of a dihedral group. If $n > 2$ then

$$Z(D_{2n}) = \begin{cases} \{1\} & \text{if } n \text{ is odd,} \\ \{1, r^{\frac{n}{2}}\} = \langle r^{\frac{n}{2}}\rangle & \text{if } n \text{ is even.} \end{cases}$$

2.3.3 Klein's Four Group. Prove that D_4 is an abelian group,

$$D_4 = \{1, s, t, st\}.$$

(It is traditionally called *Klein's Four Group.*)

2.3.4 Prove that the dihedral group D_{2n}, $n > 2$, has one class of conjugate involutions, in n is odd, and three classes, if n is even. In the latter case, one of the classes contains just one involution z and $Z(D_{2n}) = \{1, z\}$.

2.3.5 Prove that a finite group of orthogonal transformations of $\mathbb{R}^2$ is either cyclic, or a dihedral group D_{2n}.

2.3.6 If $W = D_{2n}$ is a dihedral group of orthogonal transformations of $\mathbb{R}^2$, then W has one conjugacy class of reflections, if n is odd, and two conjugacy classes of reflections, if n is even.

2.3.7 Check that the complex numbers

$$e^{2k\pi i/n} = \cos 2k\pi/n + i \sin 2k\pi/n,\ k = 0, 1, \ldots, n-1$$

in the complex plane $\mathbb{C}$ are vertices of a regular n-gon Δ. Prove that the maps

$$\begin{aligned} r : z &\mapsto z \cdot e^{2\pi i/n}, \\ t : z &\mapsto \bar{z}, \end{aligned}$$

where $\bar{\ }$ denotes the complex conjugation, generate the group of symmetries of Δ.

2.3.8 Use the idea of the proof of Theorem **??** to find the orders of the groups of symmetries of the regular tetrahedron, cube, dodecahedron.

[5] We use 'left' notation for action, so when we apply the composition st of two transformations s and t to a point, we apply t first and then s: $(st)A = s(tA)$.

2.4 Root systems

Mirrors and their normal vectors. Consider a reflection s with the mirror H. If we choose the orthogonal system of coordinates in V with the origin O belonging to H then s fixes O and thus can be treated as a linear orthogonal transformation of V. Let us take a nonzero vector α perpendicular to H then, obviously, $\mathbb{R}\alpha = H^\perp$ is the orthogonal complement of H in V, s preserves $H^\perp$ and therefore sends α to $-\alpha$. Then we can easily check that s can be written in the form

$$s_\alpha\beta = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha,$$

where (α,β) denotes the scalar product of α and β. Indeed, a direct computation shows that the formula holds when $\beta \in H$ and when $\beta = \alpha$. By the obvious linearity of the right side of the formula with respect to β, it is also true for all $\beta \in H + \mathbb{R}\alpha = V$.

Also we can check by a direct computation (left to the reader as an exercise) that, given the nonzero vector α, the linear transformation s_α is orthogonal, i.e. $(s_\alpha\beta, s_\alpha\gamma) = (\beta,\gamma)$ for all vectors β and γ. Finally, $s_\alpha = s_{c\alpha}$ for any nonzero scalar c.

Notice that reflections can be characterized as linear orthogonal transformations of $\mathbb{R}^n$ with one eigenvalue -1 and $(n-1)$ eigenvalues 1; the vector α in this case is an eigenvector corresponding to the eigenvalue -1.

Thus we have a one-to-one correspondence between the three classes of objects:

- hyperplanes (i.e. vector subspaces of codimension 1) in the Euclidean vector space V;
- nonzero vectors defined up to multiplication by a nonzero scalar;
- reflections in the group of orthogonal transformations of V.

The mirror H of the reflection s_α will be denoted by H_α. Notice that $H_\alpha = H_{c\alpha}$ for any non-zero scalar c.

Notice, finally, that orthogonal linear transformations of the Euclidean vector space V (with the origin O fixed) preserve the relations between mirrors, vectors and reflections.

Root systems. Traditionally closed systems of reflections were studied in the disguise of *root systems*. By definition, a finite set Φ of vectors in V is called a *root system* if it satisfies the following two conditions:

(1) $\Phi \cap \mathbb{R}\rho = \{\,\rho, -\rho\,\}$ for all $\rho \in \Phi$;

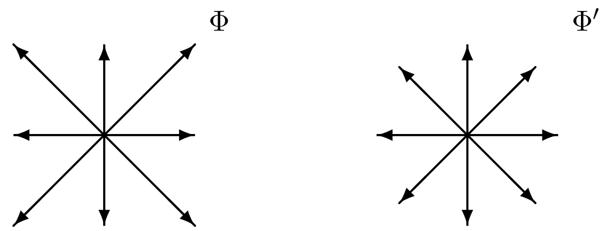


Figure 2.8: *If Φ is a root system then the vectors $\rho/|\rho|$ with $\rho \in \Phi$ form the root system Φ' with the same reflection group. We are not much interested in lengths of roots and in most cases can assume that all roots have length* 1.

(2) $s_\rho \Phi = \Phi$ for all $\rho \in \Phi$.

The following lemma is an immediate corollary of Lemma 2.1.3.

Lemma 2.4.1 *Let Σ be a finite closed system of mirrors. For every mirror H in Σ take two vectors $\pm\rho$ of length 1 perpendicular to H. Then the collection Φ of all these vectors is a root system. Vice versa, if Φ is a root system then $\{ H_\rho \mid \rho \in \Phi \}$ is a system of mirrors.*

Proof. We need only to recall that a reflection s, being an orthogonal transformation, preserves orthogonality of vectors and hyperplanes: if ρ is a vector and H is a hyperplane then $\rho \perp H$ if and only if $s\rho \perp sH$. $\square$

Also we can restate Lemma 2.2.1 in terms of root systems.

Lemma 2.4.2 *Let Φ be a root system. Then the group W generated by reflections s_ρ for $\rho \in \Phi$ is finite.*

Exercises

2.4.1 Prove, by direct computation, that the linear transformation s_α given by the formula

$$s_\alpha \beta = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha,$$

is orthogonal, that is,

$$(s_\alpha \beta, s_\alpha \beta) = (\beta, \beta)$$

for all $\beta \in V$.

2.4.2 Let Φ be a root system in the Euclidean space V and $U < V$ a vector subspace of V. Prove that $\Phi \cap U$ is a (possibly empty) root system in U.

2.4.3 Let V_1 and V_2 be two subspaces orthogonal to each other in the real Euclidean vector space V and Φ_i be a root system in V_i, $i = 1, 2$. Prove that $\Phi = \Phi_1 \cup \Phi_2$ is a root system in $V_1 \oplus V_2$; it is called the *direct sum* of Φ_1 and Φ_2 and denoted

$$\Phi = \Phi_1 \oplus \Phi_2.$$

2.4.4 We say that a group W of orthogonal transformations of V is *essential* if it acts on V without nonzero fixed points. Let Φ be a root system in V, Φ and W the corresponding system of mirrors and reflection groups. Prove that the following conditions are equivalent.

- Φ spans V.
- The intersection of all mirrors in Σ consists of one point.
- W is essential on V.

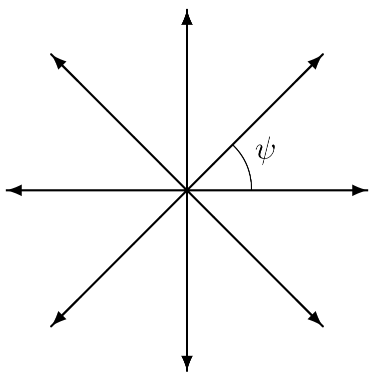
2.5 Planar root systems

We wish to begin the development of the theory of root systems with referring to the reader's geometric intuition.

Lemma 2.5.1 *If Φ is a root system in $\mathbb{R}^2$ then the angles formed by pairs of neighbouring roots are all equal.* (*See Figure* 2.9.)

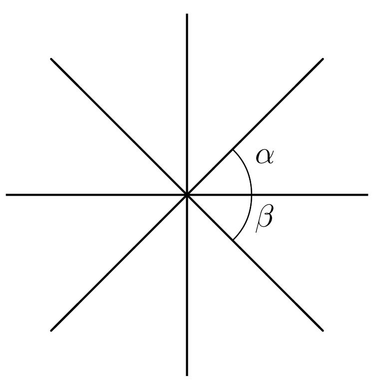
Proof of this simple result becomes self-evident if we consider, instead of roots, the corresponding system Σ of mirrors, see Figure 2.10. The mirrors in Σ cut the plane into corners (later we shall call them *chambers*), and adjacent corners, with the angles ϕ and ψ, are congruent because they are mirror images of each other. Therefore all corners are congruent. But the angle between neighbouring mirrors is exactly the angle between the corresponding roots. □

Lemma 2.5.2 *If a planar root system Φ contains $2n$ vectors, $n \geq 1$, then the reflection group $W(\Phi)$ is the dihedral group D_{2n} of order $2n$.*



The fundamental property of planar root systems: the angles ψ formed by pairs of neighbouring roots are all equal. If the root system contains $2n$ vectors then $\psi = \pi/n$ and the reflection group is the dihedral group D_{2n} of order $2n$.

Figure 2.9: A planar root system (Lemma 2.5.1).



The fact that the angles formed by pairs of neighbouring roots are all equal becomes obvious if we consider the corresponding system of mirrors: $\alpha = \beta$ because the adjacent angles are mirror images of each other.

Figure 2.10: A planar mirror system (for the proof of Lemma 2.5.1).

Proof left to the reader as an exercise. □

We see that a planar root system consisting of $2n$ vectors of equal length is uniquely defined, up to elation of $\mathbb{R}^2$. We shall denote it $I_2(n)$. Later we shall introduced planar root systems A_2 (which coincides with $I_2(3)$) as a part of series of n-dimensional root systems A_n. In many applications of the theory of reflection groups the lengths of roots are of importance; in particular, the root system $I_2(4)$ associated with the system of mirrors of symmetry of the square, comes in two versions, named B_2 and C_2, which contain 8 roots of two different lengths, see Figure 2.17 in Section 2.8. Finally, the regular hexagon gives rise to the root system of type G_2, see Figure 2.11.

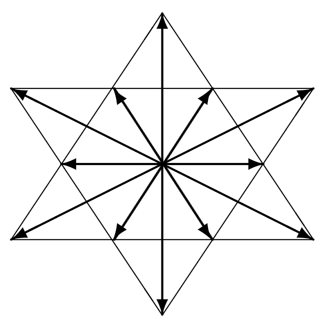


Figure 2.11: The root system G_2

Exercises

2.5.1 Prove Lemma 2.5.2.

Hint: Find a regular n-gon such that $W(\Phi)$ coincides with its symmetry group.

2.5.2 Prove that, in a root system in $\mathbb{R}^2$, the lengths of roots can take at most two values.

Hint: Use Exercise 2.3.6.

2.5.3 Describe planar root systems with 2 and 4 roots and the corresponding reflection groups.

2.5.4 Use the observation that the root system G_2 contains two subsystems of type A_2 to show that the dihedral group D_{12} contains two different subgroups isomorphic to the dihedral group D_6.

2.5.5 Crystallographic root systems. For the root systems Φ of types A_2, B_2, C_2, G_2, sketch the sets $\Lambda = \mathbb{Z}\Phi$ of points in $\mathbb{R}^2$ which are linear combinations of roots in Φ with *integer* coefficients,

$$\Lambda = \left\{ \sum_{\alpha\in\Phi} a_\alpha \alpha \mid a_\alpha \in \mathbb{Z} \right\}.$$

Observe that Λ is a subgroup of $\mathbb{R}^2$ and, moreover, a *discrete* subgroup of $\mathbb{R}^2$, that is, there is a real number $d > 0$ such that, for any $\lambda \in \Lambda$, the circle $\{ \alpha \in \mathbb{R}^2 \mid d(\alpha, \lambda) < d \}$ contains no points from Λ other than λ. We shall call root systems in $\mathbb{R}^n$ with the analogous property *crystallographic* root systems.

2.6 Positive and simple systems

Positive systems. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a linear functional. Assume that f does not vanish on roots in Φ, i.e. $f(\alpha) \neq 0$ for all $\alpha \in \Phi$. Then every root ρ in Φ is called *positive* or *negative*, according to whether $f(\rho) > 0$ or $f(\rho) < 0$. We shall write, abusing notation, $\alpha > \beta$ if $f(\alpha) > f(\beta)$. The system of all positive roots will be denoted Φ^+ and called the *positive system*. Correspondingly the *negative system* is denoted Φ^-. Obviously $\Phi = \Phi^+ \sqcup \Phi^-$.

Let Γ denotes the convex polyhedral cone spanned by the positive system Φ^+. We follow notation of Section 1.5 and call the positive roots s directed along the edges of Γ *simple* roots. The set of all simple roots is called the *simple system* of roots and denoted Π; roots in Π are called *simple* roots. It is intuitively evident that the cone Γ is generated by simple roots, see also Lemma 1.5.3. In particular, every root ϕ in Φ^+ can be written as a non-negative combination of roots in Π:

$$\phi = c_1\rho_1 + \cdots + c_m\rho_m, \quad c_i \geqslant 0, \rho_i \in \Pi.$$

Notice that the definition of positive, negative, simple systems depends on the choice of the linear functional f. We shall call a set of roots positive, negative, simple, if it is so for some functional f.

Lemma 2.6.1 *In a simple system Π, the angle between two distinct roots is non-acute: $(\alpha, \beta) \leqslant 0$ for all $\alpha \neq \beta$ in Π.*

Proof. Let P be a two-dimensional plane spanned by α and β. Denote $\Phi_0 = \Phi \cap P$. If $\gamma, \delta \in \Phi_0$ then the reflection s_γ maps δ to the vector

$$s_\gamma\delta = \delta - \frac{2(\gamma, \delta)}{(\gamma, \gamma)}\gamma$$

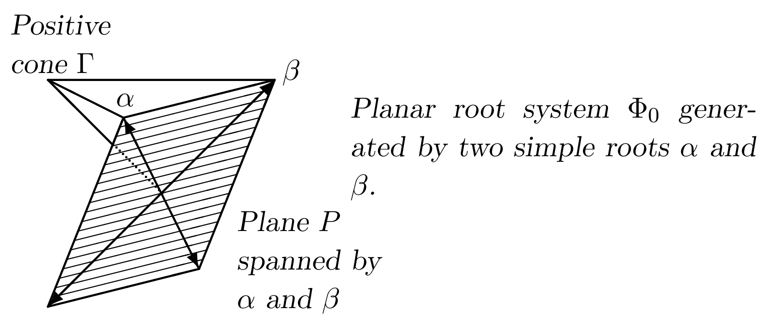


Figure 2.12: For the proof of Lemma 2.6.1

which obviously belongs to P and Φ_0. Hence every reflection s_γ for $\gamma \in \Phi_0$ obviously maps P to P and Φ_0 to Φ_0. This means that Φ_0 is a root system in P and $\Phi^+ \cap P$ is a positive system in Φ_0.

Moreover, the convex polyhedral cone Γ_0 spanned by $\Phi_0^+ = \Phi^+ \cap P$ is contained in $\Gamma \cap P$. Since α and β are obviously directed along the edges of $\Gamma \cap P$ (see also Lemma 1.5.4) and belong to Γ_0, $\Gamma_0 = \Gamma \cap P$ and α and β belong to a simple system in Φ_0, see Figure 2.12. Therefore the lemma is reduced to the 2-dimensional case, where it is self-evident, see Figure 2.13. □

Notice that our proof of Lemma 2.6.1 is a manifestation of a general principle: surprisingly many considerations in roots systems can be reduced to computations with pairs of roots.

Theorem 2.6.2 *Every simple system Π is linearly independent. In particular, every root β in Φ can be written, and in a unique way, in the form $\sum c_\alpha \alpha$ where $\alpha \in \Pi$ and all coefficients c_α are either non-negative (when $\beta \in \Phi^+$) or non-positive (when $\beta \in \Phi^-$).*

Proof. Assume, by way of contradiction, that Π is linearly dependent and

$$\sum_{\alpha \in \Pi} a_\alpha \alpha = 0$$

where some coefficient $a_\alpha \neq 0$. Rewrite this equality as $\sum b_\beta \beta = \sum c_\gamma \gamma$ where the coefficients are strictly positive and the sums are taken over disjoint subsets of Π. Set $\sigma = \sum b_\beta \beta$. Since all roots β are positive, $\sigma \neq 0$. But

$$0 \leqslant (\sigma, \sigma) = \left(\sum_\beta b_\beta \beta, \sum_\gamma c_\gamma \gamma \right) = \sum_\beta \sum_\gamma b_\beta c_\gamma (\beta, \gamma) \leqslant 0,$$

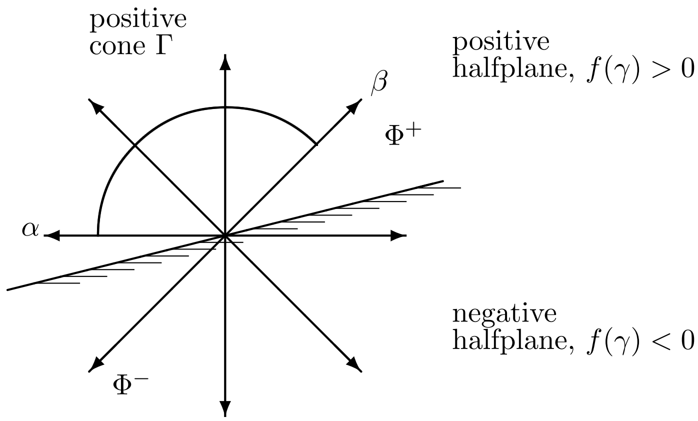


Figure 2.13: *For the proof of Lemma* 2.6.1. *In the* 2*-dimensional case the obtuseness of the simple system is obvious: the roots* α *and* β *are directed along the edges of the convex cone spanned by* Φ^+ *and the angle between* α *and* β *is at least* $\pi/2$.

because all individual scalar products (β, γ) are non-positive by Lemma 2.6.1. Therefore $\sigma = 0$, a contradiction. □

Corollary 2.6.3 *All simple systems in* Φ *contain an equal number of roots.*

Proof. Indeed, it follows from Theorem 2.6.2 that a simple system is a maximal linearly independent subset of Φ. □

The number of roots in a simple system of the root system Φ is called the *rank* of Φ and denoted $\mathrm{rk}\,\Phi$. The subscript n in the standard notation for root systems A_n, B_n, etc. (which will be introduced later) refers to their ranks.

Exercises

2.6.1 Prove that, in a planar root system $\Phi \subset \mathbb{R}^2$, all positive (correspondingly, simple) systems are conjugate under the action of the reflection group $W = W(\Phi)$.

2.7 Root system A_{n-1}.

Permutation representation of Sym_n**.** Let V be the real vector space $\mathbb{R}^n$ with the standard orthonormal basis $\epsilon_1, \ldots, \epsilon_n$ and the corresponding coordinates $x_1, \ldots, x_n$.

The group $W = Sym_n$ acts on V in the natural way, by permuting the n vectors $\epsilon_1, \ldots, \epsilon_n$:

$$w\epsilon_i = \epsilon_{wi},$$

which, obviously, induces an action of W on Φ. The action of the group $W = Sym_n$ on $V = \mathbb{R}^n$ preserves the standard scalar product associated with the orthonormal basis $\epsilon_1, \ldots, \epsilon_n$. Therefore W acts on V by orthogonal transformations.

In its action on V the transposition $r = (ij)$ acts as the reflection in the mirror of symmetry given by the equation $x_i = x_j$.

Lemma 2.7.1 *Every reflection in W is a transposition.*

Proof. The cycle $(i_1 \cdots i_k)$ has exactly one eigenvalue 1 when restricted to the subspace $\mathbb{R}\epsilon_{i_1} \oplus \cdots \oplus \mathbb{R}\epsilon_{i_k}$, with the eigenvector $\epsilon_{i_1} + \cdots + \epsilon_{i_k}$. It follows from this observation that the multiplicity of eigenvalue 1 of the permutation $w \in \mathrm{Sym}_n$ equals the number of cycles in the cycle decomposition of w (we have to count also the trivial one-element cycles of the form (i)). If w is a reflection, then the number of cycles is $n-1$, hence w is a transposition. $\square$

Regular simplices. The convex hull Δ of the points $\epsilon_1, \ldots, \epsilon_n$ is the convex polytope defined by the equation and inequalities

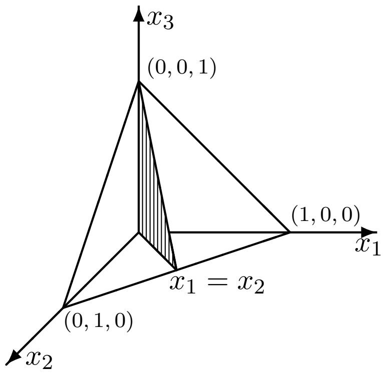
$$x_1 + \cdots + x_n = 1, \; x_1 \geqslant 0, \ldots, x_n \geqslant 0.$$

Since the group $W = \mathrm{Sym_n}$ permutes the vertices of Δ, it acts as a group of symmetries of Δ, $W \leqslant \mathrm{Sym}\,\Delta$. We wish to prove that actually $W = \mathrm{Sym}\,\Delta$. Indeed, any symmetry s of Δ acts on the set of vertices as some permutation $w \in \mathrm{Sym}_n$, hence the symmetry $s^{-1}w$ fixes all the vertices $\epsilon_1, \ldots, \epsilon_n$ of Δ and therefore is the identity symmetry.

The polytope Δ is called the *regular* $(n-1)$*-simplex.* When $n = 3$, Δ is an equilateral triangle lying in the plane $x_1 + x_2 + x_3 = 1$ (see Figure 2.14), and when $n = 4$, Δ is a regular tetrahedron lying in the 3-dimensional affine Euclidean space $x_1 + x_2 + x_3 + x_4 = 1$.

The root system A_{n-1}. We shall introduce the root system Φ of type A_{n-1}, as the system of vectors in $V = \mathbb{R}^n$ of the form $\epsilon_i - \epsilon_j$, where $i, j = 1, 2, \ldots, n$ and $i \neq j$. Notice that Φ is invariant under the action of $W = \mathrm{Sym}_n$ on V.

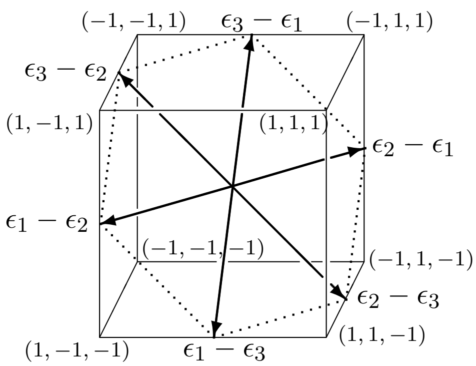
In its action on V the transposition $r = (ij)$ acts as the reflection in the mirror of symmetry perpendicular to the root $\rho = \epsilon_i - \epsilon_j$. Hence Φ is a root system. Since the symmetric group is generated by transpositions,



The transposition (12) *acts on* $\mathbb{R}^3$ *as the reflection in the mirror* $x_1 = x_2$ *and as a symmetry of the equilateral triangle with the vertices*

$$(1,0,0),\ (0,1,0),\ (0,0,1).$$

Figure 2.14: Sym_n is the group of symmetries of the regular simplex.



The root system $\{\, \epsilon_i - \epsilon_j \mid i \neq j \,\}$ *of type* A_2 *lies in the hyperplane* $x_1 + x_2 + x_3 = 0$ *which cuts a regular hexagon in the unit cube* $[-1,1]^3$.

Figure 2.15: Root system of type A_2.

$W = W(\Phi)$ is the corresponding reflection group, and the mirror system Σ consists of all hyperplanes $x_i = x_j$, $i \neq j$, $i,j = 1,\dots,n$.

Notice that the group W is not essential for V; indeed, it fixes all points in the 1-dimensional subspace $\mathbb{R}(\epsilon_1 + \cdots + \epsilon_n)$ and leaves invariant the $(n-1)$-dimensional linear subpace U defined by the equation $x_1 + \cdots + x_n = 0$. It is easy to see that $\Phi \subset U$ spans U. In particular, the rank of the root system Φ is n, which justifies the use, in accordance with our convention, of the index $n-1$ in the notation A_{n-1} for it.

The standard simple system. Take the linear functional

$$f(x) = x_1 + 2x_2 + \cdots + nx_n.$$

Obviously f does not vanish on roots, and the corresponding positive system has the form

$$\Phi^+ = \{\, \epsilon_i - \epsilon_j \mid j < i \,\}.$$

The set of positive roots

$$\Pi = \{\, \epsilon_2 - \epsilon_1, \epsilon_3 - \epsilon_2, \ldots, \epsilon_n - \epsilon_{n-1} \,\}$$

is linearly independent and every positive root is obviously a linear combination of roots in Π with nonnegative coefficients: for $i > j$,

$$\epsilon_i - \epsilon_j = (\epsilon_i - \epsilon_{i-1}) + \cdots + (\epsilon_{j+1} - \epsilon_j),$$

Therefore Π is a simple system. It is called the *standard simple system* of the root system A_{n-1}.

Action of Sym_n on the set of all simple systems. The following result is a partial case of Theorem 3.5.1. But the elementary proof given here is instructive on its own.

Lemma 2.7.2 *The group $W = \mathrm{Sym}_n$ acts simply transitively on the set of all positive* (*resp. simple*) *systems in Φ.*

Proof. Since there is a natural one-to-one correspondence between simple and positive systems, it is enough to prove that W acts simply transitivly on the set of positive systems in Φ.

Let f be an arbitrary linear functional which does not vanish on Φ, that is, $f(\epsilon_i - \epsilon_j) \neq 0$ for all $i \neq j$. Then all the values

$$f(\epsilon_1), \ldots, f(\epsilon_n)$$

are different and we can list them in the strictly increasing order:

$$f(\epsilon_{i_1}) < f(\epsilon_{i_2}) < \ldots < f(\epsilon_{i_n}).$$

Now consider the permutation w given, in the column notation, as

$$w = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ i_1 & i_2 & \cdots & i_{n-1} & i_n \end{pmatrix}.$$

Thus the functional f defines a new ordering, wich we shall denote as $\leqslant^w$, on the set $[n]$:

$$j \leqslant^w i \text{ if and only if } f(\epsilon_j) \leqslant f(\epsilon_i).$$

If we look again at the table for w we see that above any element i in the bottom row lies, in the upper row, the element $w^{-1}i$. Thus

$$i \leqslant^w j \text{ if and only if } w^{-1}i \leqslant w^{-1}j.$$

Notice also that the permutation w and the associated ordering $\leqslant^w$ of $[n]$ uniquely determine each other[6].

Now consider the positive system Φ_0^+ defined by the functional f,

$$\Phi_0^+ = \{\, \epsilon_i - \epsilon_j \mid f(\epsilon_i - \epsilon_j) > 0 \,\}.$$

We have the following chain of equivalences:

$$\begin{aligned}
\epsilon_i - \epsilon_j \in \Phi_0^+ \quad &\text{iff} \quad f(\epsilon_j) < f(\epsilon_i) \\
&\text{iff} \quad j <^w i \\
&\text{iff} \quad w^{-1}j < w^{-1}i \\
&\text{iff} \quad \epsilon_{w^{-1}i} - \epsilon_{w^{-1}j} \in \Phi^+ \\
&\text{iff} \quad w^{-1}(\epsilon_i - \epsilon_j) \in \Phi^+ \\
&\text{iff} \quad \epsilon_i - \epsilon_j \in w\Phi^+.
\end{aligned}$$

This proves that $\Phi_0^+ = w\Phi^+$ and also that the permutation w is uniquely determined by the positive system Φ_0^+. Since Φ^+, by its construction from an arbitrary functional, represents an arbitrary positive system in Φ, the group W acts on the set of positive systems in Φ simply transitively. □

Exercises

2.7.1 Make a sketch of the root systems $A_1 \oplus A_1$ in $\mathbb{R}^2$ and $A_1 \oplus A_1 \oplus A_1$ in $\mathbb{R}^3$.

2.7.2 Check that, when we take the intersections of the mirrors of reflections in $W = \mathrm{Sym}_n$ to the subspace $x_1 + \cdots + x_n = 0$ of $\mathbb{R}^n$, the resulting system of mirrors can be geometrically described as the system of mirrors of symmetry of the regular $(n-1)$-simplex with the vertices

$$\delta_i = \epsilon_i - \frac{1}{n}(\epsilon_1 + \cdots + \epsilon_n), \quad i = 1, \ldots, n.$$

2.7.3 An orthogonal transformation of the Euclidean space $\mathbb{R}^n$ is called a *rotation* if its determinant is 1. For a polytope Δ, denote by $\mathrm{Rot}\,\Delta$ the subgroup of $\mathrm{Sym}\,\Delta$ formed by all rotations. We know that the group of symmetries $\mathrm{Sym}\,\Delta$ of the regular tetrahedron Δ in $\mathbb{R}^3$ is isomorphic to Sym_4; prove that $\mathrm{Rot}\,\Delta$ is the alternating group Alt_4.

[6] In the nineteenth century the orderings, or rearrangements, of $[n]$ were called *permutations*, and the permutations in the modern sense, i.e. maps from $[n]$ to $[n]$, were called *substitutions*. These are two aspects, 'passive' and 'active', of the same object. We shall see later that they correspond to treating a permutation as an element of a reflection group Sym_n or an element of the Coxeter complex for Sym_n.

2.8 Root systems of type C_n and B_n

Hyperoctahedral group. Let

$$[n] = \{1, 2, \dots, n\} \text{ and } [n]^* = \{1^*, 2^*, \dots, n^*\}.$$

Define the map $* : [n] \longrightarrow [n]^*$ by $i \mapsto i^*$ and the map $* : [n]^* \longrightarrow [n]$ by $(i^*)^* = i$. Then $*$ is an involutive permutation[7] of the set $[n] \sqcup [n]^*$.

Let W be the group of all permutations of the set $[n] \sqcup [n]^*$ which commute with the involution *, i.e. a permutation w belongs to W if and only if $w(i^*) = w(i)^*$ for all $i \in [n] \sqcup [n]^*$. We shall call permutations with this property *admissible*. The group W is known under the name of *hyperoctahedral group BC_n*. It is easy to see that W is isomorphic to the group of symmetries of the n-cube $[-1, 1]^n$ in the n-dimensional real Euclidean space $\mathbb{R}^n$. Indeed, if $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ is the standard orthonormal basis in $\mathbb{R}^n$, we set, for $i \in [n]$, $\epsilon_{i^*} = -\epsilon_i$. Then we can define the action of W on $\mathbb{R}^n$ by the following rule: $w\epsilon_i = \epsilon_{wi}$. Since w is an admissible permutation of $[n] \sqcup [n]^*$, the linear transformation is well-defined and orthogonal. Also it can be easily seen that W is exactly the group of all orthogonal transformations of $\mathbb{R}^n$ preserving the set of vectors $\{\pm\epsilon_1, \pm\epsilon_2, \dots, \pm\epsilon_n\}$ and thus preserving the cube $[-1, 1]^n$. Indeed, the vectors $\pm\epsilon_i$, $i \in [n]$, are exactly the unit vectors normal to the $(n-1)$-dimensional faces of the cube (given, obviously, by the linear equations $x_i = \pm 1$, $i = 1, 2, \dots, n$).

The name 'hyperoctahedral' for the group W is justified by the fact that the group of symmetries of the n-cube coincides with the group of symmetries of its dual polytope, whose vertices are the centers of the faces of the cube. The dual polytope for the n-cube is known under the name of *n-cross polytope* or *n-dimensional hyperoctahedron* (see Figure 2.16).

Admissible orderings. We shall order the set $[n] \sqcup [n]^*$ in the following way:

$$n^* < n-1^* < \dots < 2^* < 1^* < 1 < 2 < \dots < n-1 < n.$$

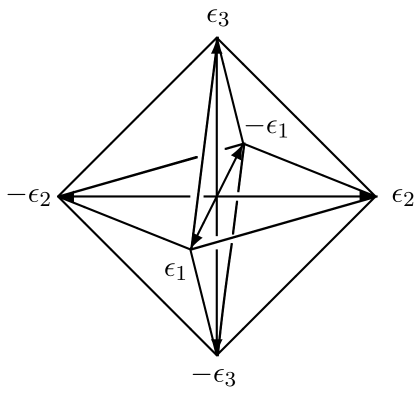
If now $w \in W$ then we define a new ordering $\leqslant^w$ of the set $[n] \sqcup [n]^*$ by the rule

$$i \leqslant^w j \textit{ if and only if } w^{-1}i \leqslant w^{-1}j.$$

Orderings of the form $\leqslant^w$, $w \in W$, are called *admissible* orderings of the set $[n] \sqcup [n]^*$. They can be characterized by the following property:

> *an ordering $\prec$ on $[n] \sqcup [n]^*$ is admissible if and only if from $i \prec j$ it follows that $j^* \prec i^*$.*

[7] That is, a permutation of order 2.



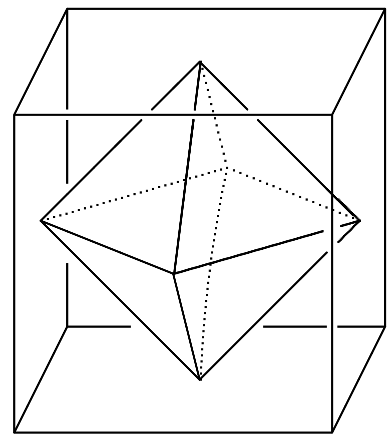


Figure 2.16: *Hyperoctahedron* (*'octahedron' in dimension* $n = 3$) *or* n*-cross polytope is the convex hull of the points* $\pm\epsilon_i$, $i = 1, \ldots, n$ *in* $\mathbb{R}^n$ (*picture* (a)). *Obviously the hyperoctahedron is the dual polytope to the unit cube* (*picture* (b)).

Vice versa, if $\prec$ is an admissible ordering, then the permutation

$$w = \begin{pmatrix} n^* & (n-1)^* & \ldots & 1^* & 1 & \ldots & n-1 & n \\ j_1 & j_2 & \ldots & j_n & j_{n+1} & \ldots & j_{2n-1} & j_{2n} \end{pmatrix}$$

where

$$j_1 \prec j_2 \prec \ldots \prec j_{2n-1} \prec j_{2n},$$

is admissible and the ordering $\prec$ coincides with $\leqslant^w$.

Root systems C_n and B_n. Let ϵ_i, $i \in [n]$, be the standard orthonormal basis in $\mathbb{R}^n$, and set $\epsilon_{i^*} = -\epsilon_i$ for $i^* \in [n]^*$. This defines the vectors ϵ_j for all $j \in [n] \sqcup [n]^*$. Now the root system Φ of type C_n is formed by the vectors $2\epsilon_j$, $j \in [n] \sqcup [n]^*$ (called *long roots*), together with the vectors $\epsilon_{j_1} - \epsilon_{j_2}$, where $j_1, j_2 \in [n] \sqcup [n]^*$, $j_1 \neq j_2$ or j_2^* (called *short roots*). Written in the standard basis $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, the roots take the form $\pm 2\epsilon_i$ or $\pm\epsilon_i \pm \epsilon_j$, $i, j = 1, 2, \ldots, n$, $i \neq j$. Notice that both short and long roots can be written as $\epsilon_j - \epsilon_i$ for some $i, j \in [n] \sqcup [n]^*$.

It is easy to see that when ρ is one of the long roots $\pm 2\epsilon_i$, $i \in [n]$, then s_ρ is the linear transformation corresponding to the element (i, i^*) of W in its canonical representation. Analogously, if $\rho = \epsilon_i - \epsilon_j$, is a short root (recall that we use the convention $\epsilon_{i^*} = -\epsilon_i$ for $i \in [n]$), then the reflection s_r corresponds to the admissible permutation $(i, j)(i^*, j^*)$. Moreover, one can easily check (for example, by computing the eigenvalues of admissible permutations from W in their action on $\mathbb{R}^n$) that every reflection in the

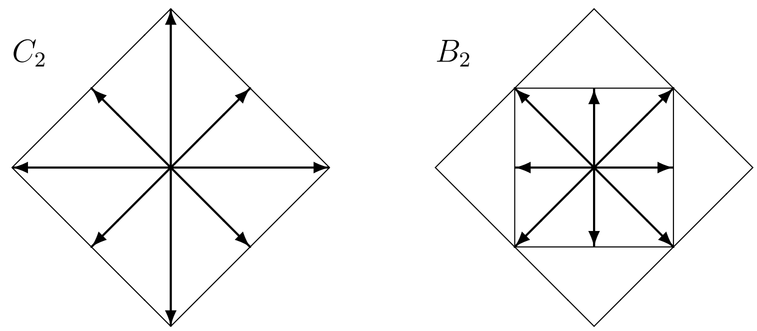


Figure 2.17: Root systems B_2 and C_2.

group of the symmetries of the unit cube $[-1, 1]^n$ is of one of these two types.

Now we see that use of the name 'root system' in regard to the set Φ is justified.

The root system

$$B_n = \{ \pm\epsilon_i \pm \epsilon_j ,\ \pm\epsilon_i \mid i, j = 1, 2 \ldots, n,\ i \neq j \}$$

differs from C_n only in lengths of roots (see Figure 2.17) and has the same reflection group BC_n. Therefore in the sequel we shall deal only with the root system C_n.

Action of W on Φ. Now consider the linear functional

$$f(x) = x_1 + 2x_2 + 3x_3 + \cdots + nx_n.$$

It is easy to see that a root $\epsilon_i - \epsilon_j$ is positive with respect to f if, in the ordering

$$n^* < n-1^* < \ldots < 1^* < 1 < 2 < \ldots < n$$

of the set $[n] \sqcup [n]^*$, we have $i > j$. The system of positive roots Φ^+ associated with f is called the *standard positive system of roots*. The set

$$\Pi = \{2\epsilon_1, \epsilon_2 - \epsilon_1, \ldots, \epsilon_n - \epsilon_{n-1}\}$$

is obviously the simple system of roots contained in Φ^+.

If now

$$j_1 <^w j_2 <^w \ldots <^w j_{2n-1} <^w j_{2n},$$

is an admissible ordering of $[n] \sqcup [n]^*$, then the vectors $\epsilon_{j_{n+1}}, \epsilon_{j_{n+2}}, \ldots, \epsilon_{j_{2n}}$ form a basis in $\mathbb{R}^n$. Let $y_1, y_2, \ldots, y_n$ be the coordinates with respect to this basis and $f(y) = y_1 + 2y_2 + 3y_3 + \cdots + ny_n$. Then, obviously, f does not vanish on roots in Φ, and, for a root $\epsilon_j - \epsilon_i$ in Φ, the inequality $f(\epsilon_j - \epsilon_i) > 0$

is equivalent to $i \leqslant^w j$. Thus the system of positive roots associated with f coincides with the system

$$w\Phi^+ = \{\epsilon_j - \epsilon_i \mid i \leqslant^w j\}$$

obtained from the standard system Φ^+ of positive roots by the action of the element w. Obviously, the simple system of roots contained in Φ^+ is exactly $w\Pi$.

If now Π' is an arbitrary simple system of roots arising from an arbitrary linear function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ not vanishing on roots in Φ then the following objects are uniquely determined by our choice of Π':

- the system of positive roots $\Phi^{+\prime}$, which can be defined in two equivalent ways: as the set of all roots which are non-negative linear combinations of roots from Π', and as the set $\{r \in \Phi \mid f(r) > 0\}$;
- the (obviously admissible) ordering $\prec$ on J defined by the rule: $i \prec j$ if and only if $f(\epsilon_i) \leqslant f(\epsilon_j)$.

In particular, we immediately have the following observation (which is a partial case of a more general result about conjugacy of simple system of roots for arbitrary finite reflection groups, Theorem 3.5.5).

Proposition 2.8.1 *Any two simple systems in the root system Φ of type B_n or C_n are conjugate under the action of W. Moreover, the reflection group W is simply transitive in its action on the set of simple systems in Φ.*

Exercises.

2.8.1 Prove that every reflection in BC_n has the form (ii^*) or $(ij)(i^*j^*)$.

2.8.2 Prove that the reflection group of type BC_2 is isomorphic to the dihedral group D_8.

2.8.3 THE GROUP OF SYMMETRIES OF THE CUBE. Observe that the group $W = BC_3$ of symmetries of the cube $\Delta = [-1,1]^3$ contains the involution z which sends every vertex of the cube to its opposite.

1. Check that $\det z = -1$, so that z to the group $R = \mathrm{Rot}\,\Delta$ of rotations of the cube.
2. Prove that $z \in Z(W)$.
3. Prove that the group R acts faithfully on the set D of 4 diagonals of the cube Δ, that is, the segments connecting the opposite vertices of the cube. Moreover, every permutation of diagonals is the result of action of a rotation of the cube. Hence $R \simeq \mathrm{Sym}_4$.

4. Prove that $W = \langle z \rangle \times R$.
5. Prove that $\langle z \rangle = Z(W)$.
6. Prove that the symmetries of the cube which send every 2-dimensional face of the cube into itself or the opposite face form a normal abelian subgroup $E < W$ of order 8. Prove further that $W/E \simeq \mathrm{Sym}_3$ and that actually $W = E \rtimes T$ for some subgroup $T \simeq \mathrm{Sym}_3$.

2.8.4 Important root subsystems.

Prove that

1. the set Θ of roots $\{ \pm\epsilon_i \mid i = 1, \ldots, n \}$ is a root system of type $A_1 + \cdots + A_1$ (n summands);
2. the intersection Ψ of Φ with the hyperplane $x_1 + \cdots + x_n = 0$ is a root system of type A_{n-1}.

2.8.5 The structure of the hyperoctahedral group. Use Exercise 2.8.4 to show that if E and T are the reflection groups corresponding to the systems of roots Θ and Ψ then

1. $E \simeq \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors);
2. $E \lhd W$;
3. $T \simeq \mathrm{Sym}_n$;
4. $W = E \rtimes T$.

2.8.6 (R. Sandling) Prove that

$$\mathrm{Sym}\,[-1,1]^n = \{ w \in \mathrm{GL}_n(\mathbb{R}^n) \mid w([-1,1]^n) = [-1,1]^n \},$$

i.e. linear transformations preserving the cube are in fact orthogonal.

2.9 The root system D_n

By definition,

$$D_n = \{ \pm\epsilon_i \pm \epsilon_j \mid i, j = 1, 2, \ldots, n, i \neq j \};$$

thus D_n is a subsystem of the root system C_n.

The system

$$\Pi = \{ \epsilon_1 + \epsilon_2, \epsilon_2 - \epsilon_1, \epsilon_3 - \epsilon_2, \ldots, \epsilon_n - \epsilon_{n-1} \}$$

is a simple system in Φ.

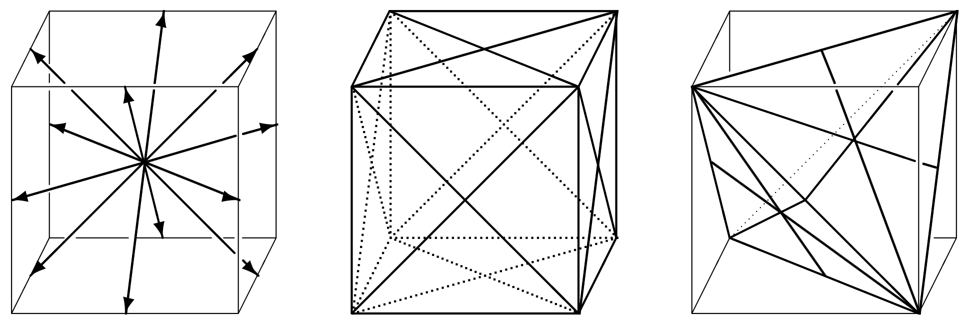


Figure 2.18: *Shown are the root system D_3 inscribed into the unit cube $[-1,1]^3$ (on the left), and the corresponding mirror system (shown in the middle by intersections with the surface of the cube and the tetrahedron inscribed in the cube). Comparing the last two pictures we see that the mirror system of type D_3 is isometric to the mirror system of type A_3.*

Exercises

2.9.1 Describe explicitly an isometry between the root systems

$$D_3 = \{ \pm\epsilon_i \pm \epsilon_j \mid i,j = 1,2,3, i \neq j \}$$

and

$$A_3 = \{ \epsilon_i - \epsilon_j \mid i,j = 1,2,3,4, i \neq j \}$$

(see Figure 2.18).

2.9.2 Sketch the root system D_2; you will see that it consists of two orthogonal pairs of vectors, each forming the 1-dimensional system A_1. Thus $D_2 = A_1 + A_1$.

Chapter 3

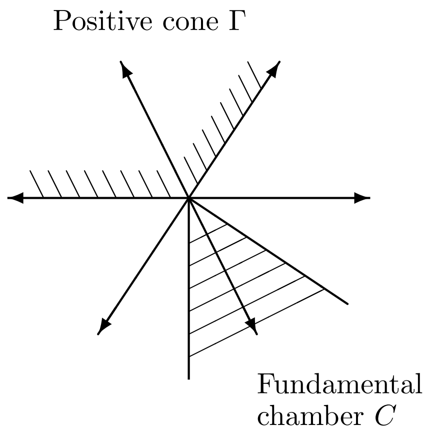
Coxeter Complex

3.1 Chambers

Chambers. Consider the system Σ of all mirrors of reflections s_ρ for $\rho \in \Phi$. Of course, this a hyperplane arrangement in the sense of Section 1.2, and we shall freely use the relevant terminology. In particular, chambers of Σ are open polyhedral cones—connected components of $V \smallsetminus \bigcup_{H\in\Sigma} H$. The closures of these cones are called *closed chambers*. Facets of chambers (i.e. faces of maximal dimension) are *panels* and mirrors in Σ as *walls*. Notice that every panel belongs to a unique wall. To fully appreciate this architectural terminology (introduced by J. Tits), imagine a building built out of walls of double-sided mirrors. Two chambers are *adjacent* if they have a panel in common. Notice that every chamber is adjacent to itself.

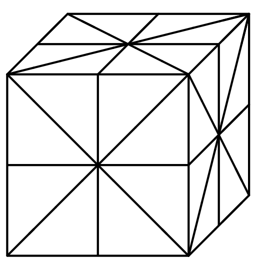
Theorem 3.1.1 *Every chamber C has the form $\bigcap_{\rho\in\Pi} V_\rho^-$ for some simple system Π. Every panel of C belongs to one of the walls H_ρ for a root $\rho \in \Pi$. Vice versa, if $\rho \in \Pi$ then $H_\rho \cap \overline{C}$ is a panel of C.*

Proof. Take any vector γ in the chamber C and consider the linear functional $f(\lambda) = -(\gamma, \lambda)$. Since γ does not belong to any mirror H_α in Σ, the functional f does not vanish on roots in Φ. Therefore the condition $f(\alpha) > 0$ determines a positive system Φ^+ and the corresponding simple system Π. Now consider the cone C' defined by the inequalities $(\lambda, \rho) < 0$ for all $\rho \in \Pi$. Obviously $\gamma \in C'$ and therefore $C \subseteq C'$. If $C \neq C'$ then some hyperplane H_α, $\alpha \in \Phi$, bounds C and intersects C' nontrivially. But $\alpha = \sum c_\rho \rho$ where all c_ρ are all non-negative or all non-positive, and $(\gamma, \alpha) = \sum c_\rho(\gamma, \rho)$ cannot be equal 0. This contradiction shows that $C = C'$. The closure $\overline{C}$ of C is defined by the inequalities $(\lambda, \rho) \leqslant 0$ for $\rho \in \Pi$ which is equivalent to $(\lambda, \alpha) \leqslant 0$ for $\alpha \in \Gamma$. Therefore the cone



The fundamental chamber C is defined as the interior of the cone dual to the positive cone Γ, i.e. as the set of vectors λ such that $(\lambda, \gamma) < 0$ for all $\gamma \in \Gamma$.

Figure 3.1: The fundamental chamber.



The Coxeter complex of type BC_3 is formed by all the mirrors of symmetry of the cube; here they are shown by their lines of intersection with the faces of the cube.

Figure 3.2: The Coxeter complex BC_3.

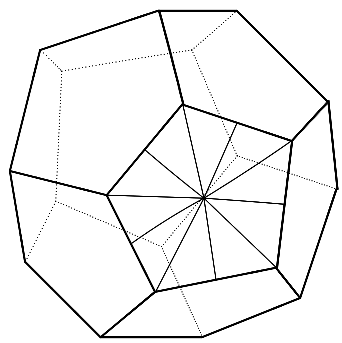
$\overline{C}$ is dual to the positive cone Γ (see Figure 3.1) and every facet of C is perpendicular to some edge of Γ, and vice versa. In particular, every panel of C belongs to the wall H_ρ for some simple root $\rho \in \Pi$.

The same argument works in the reverse direction: if Π is any simple system then, since Π is linearly independent, we can find a vector γ such that $(\gamma, \rho) < 0$ for all $\rho \in \Pi$. Then $(\gamma, \alpha) \neq 0$ for all roots $\alpha \in \Phi$ and the chamber C containing γ has the form $C = \bigcap_{\rho \in \Pi} V_\rho^-$. $\square$

The set of all chambers associated with the root system Φ is called the *Coxeter complex* and will be denoted by $\mathcal{C}$. See, for example, Figures 3.2 and 3.3. If Π is a simple system then the corresponding chamber is called the *fundamental chamber* of $\mathcal{C}$.

The following lemma is an immediate consequence of Lemma 1.2.3

Lemma 3.1.2 *The union of two distinct adjacent closed chambers is con-*



In the case that Σ is the system of mirrors of symmetry of a regular polytope Δ, the Coxeter complex is basically the c subdivision of the faces of Δ by the mirrors of symmetries of faces (here shown only on one face of the dodecahedron Δ).

Figure 3.3: *Chambers and baricentric subdivision*

The symmetry group of the tetrahedron acts on its 4 vertices as the symmetric group Sym_4. The reflections in the walls of the fundamental chamber are the transpositions (12), (23) and (34). Therefore they generate Sym_4.

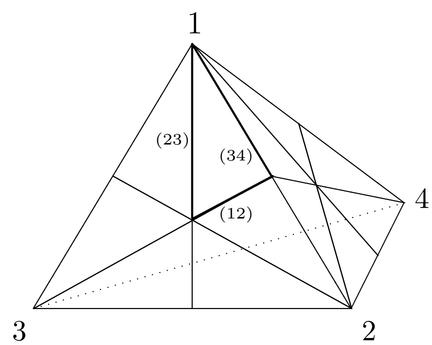


Figure 3.4: Generation by simple reflections (Theorem 3.2.1).

vex.

3.2 Generation by simple reflections

Simple reflections. Let $\Pi = \{ \rho_1, \ldots, \rho_n \}$ be a simple system of roots. The corresponding reflections $r_i = s_{\rho_i}$ are called *simple reflections.*

Theorem 3.2.1 *The group W is generated by simple reflections.*

Proof. Set $W' = \langle r_1, \ldots, r_n \rangle$. We shall prove first that

the group W' is transitive in its action on $\mathcal{C}$.

Proof of the claim. The fundamental chamber C is bounded by panels lying on the mirrors of the simple reflections $r_1, \ldots, r_n$. Therefore the neighbouring chambers (i.e. the chambers sharing a common mirror with C) can be obtained from C by reflections in these mirrors and equal $r_1C, \ldots, r_nC$.

Let now $w \in W'$, then the panels of the chamber wC belong to the mirrors of reflections $wr_1w^{-1}, \ldots, wr_nw^{-1}$. If D is a chamber adjacent to wC then it can be obtained from wC by reflecting wC in the common mirror, hence $D = wr_iw^{-1} \cdot wC = wr_iC$ for some $i = 1, \ldots, n$. Notice that $wr_i \in W'$. We can proceed to move from a chamber to an adjacent one until we present all chambers in $\mathcal{C}$ in the form wC for appropriate elements $w \in W$. $\square$

We can now complete the proof. If $\alpha \in \Phi$ is any root and s_α the corresponding reflection then the wall H_α bounds some chamber D. We know that $D = wC$ for some $w \in W'$. The fundamental chamber C is bounded by the walls H_{ρ_i} for simple roots ρ_i (Theorem 3.1.1) and therefore the wall H_α equals wH_{ρ_i} for some simple root ρ_i. Thus $s_\alpha = wr_iw^{-1}$ belongs to W'. Since the group W is generated by reflections s_α we have $W = W'$. $\square$ $\square$

In the course of the proof we have obtained one more important result:

Corollary 3.2.2 *The action of W on $\mathcal{C}$ is transitive.*

This observation will be later incorporated into Theorem 3.5.1.

Exercises

3.2.1 Use Theorem 3.2.1 to prove the (well-known) fact that the symmetric group Sym_n is generated by transpositions $(12), (23), \ldots, (n-1, n)$ (see Figure 3.4).

3.2.2 Prove that the reflections

$$r_1 = (12)(1^*2^*), \ldots, r_{n-1} = (n-1, n)(n-1^*, n^*),\ r_n = (n, n^*)$$

generate the hyperoctahedral group BC_n.

3.3 Foldings

Given a non-zero vector $\alpha \in V$, the hyperplane $H_\alpha = \{\, \gamma \in V \mid (\gamma, \alpha) = 0 \,\}$ cuts V in two subspaces

$$V_\alpha^+ = \{\, \gamma \mid (\gamma, \alpha) \geqslant 0 \,\} \text{ and } V_\alpha^- = \{\, \gamma \mid (\gamma, \alpha) \leqslant 0 \,\}$$

intersecting along the common hyperplane H_α. The *folding* in the direction of α is the map f_α defined by the formula

$$f_\alpha(\beta) = \begin{cases} \beta & \text{if } (\beta, \alpha) \geqslant 0 \\ s_\alpha\beta & \text{if } (\beta, \alpha) < 0 \end{cases} .$$

In the 2-dimensional case, a folding is exactly what its name suggests: the plane is being folded on itself like a sheet of paper.

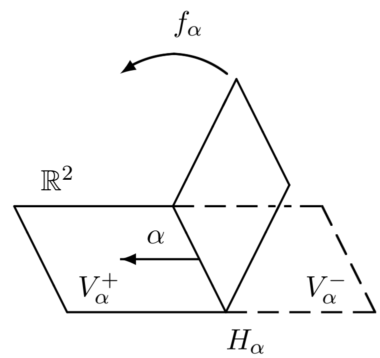


Figure 3.5: Folding

Thus f_α fixes all points in V_α^+ and maps V_α^- onto V_α^+ symmetrically (see Figure 3.5). Notice that f_α is an idempotent map, i.e. $f_\alpha f_\alpha = f_\alpha$. The folding $f_{-\alpha}$ is called the *opposite* to f_α. The reflection s_α is made up of two foldings f_α and $f_{-\alpha}$:

$$s_\alpha = f_\alpha|_{V_\alpha^+} \cup f_{-\alpha}|_{V_\alpha^-}\ .$$

We say that a folding f *covers* a subset $X \subset V$ if $X \subseteq f(V)$.

By definition, a folding of the chamber complex $\mathcal{C}$ is a folding along one of its walls.

Proposition 3.3.1 *A folding f of $\mathcal{C}$ sends chambers to chambers and preserves adjacency: if C and D are two adjacent chambers then their images $f(C)$ and $f(D)$ are also adjacent. (Remember that, by definition of adjacency, this includes the possibility that $f(C) = f(D)$.)*

Exercises

3.3.1 When you fold a sheet of paper, why is the line along which it is folded straight?

3.3.2 There are three foldings of the chamber complex BC_2 such that their composition maps the chamber complex onto one of its chambers. What is the minimal number of foldings needed for folding the chamber complex BC_3 onto one chamber?

3.4 Galleries and paths

Galleries. Given two chambers C and D, we can always find a sequence G of chambers $C = C_0, C_1, \ldots, C_{l-1}, C_l$ such that every two consequtive chambers C_{i-1} and C_i are adjacent. We shall call G a *gallery* connecting the chambers C and D. Notice that our definition of adjacency allows that two adjacent chambers coincide. This means that we also allow repetition of chambers in a gallery: it could happen that $C_{i-1} = C_i$. We shall say in this situation that the gallery *stutters* at chamber C_i. The number l will be called the *length* of the gallery G.

Notice that if s_i is the reflection in a common wall of two adjacent chambers C_{i-1} and C_i then either $C_i = s_i C_{i-1}$ or $C_i = C_{i-1}$.

Given $w \in W$, we wish to describe a canonical way of connecting the fundamental chamber C and the chamber $D = wC$ by a gallery. We know that W is generated by the fundamental reflections $r_1, \ldots, r_n$, i.e. the reflections in the walls of the fundamental chamber C. The minimal number l such that w is the product of some l fundamental reflections is called the *length* of w and denoted as $l(w)$.

Let $w = r_{i_1} \cdots r_{i_l}$. We leave to the reader to check the following group theoretical identity: since all r_i are involutions,

$$r_{i_1} \cdots r_{i_l} = r_{i_l}^{r_{i_{l-1}} \cdots r_{i_1}} \cdot r_{i_{l-1}}^{r_{i_{l-2}} \cdots r_{i_1}} \cdots r_{i_2}^{r_{i_1}} \cdot r_{i_1}.$$

Denote $s_j = r_{i_j}^{r_{j-1} \cdots r_{i_1}}$, then $w = s_l \cdots s_1$ and, moreover,

$$s_j \cdots s_1 = r_{i_1} \cdots r_{i_j} \ \text{ for } j = 1, \ldots, l.$$

Define by induction $C_0 = C$ and, for $i = 1, \ldots, l$, $C_j = s_j C_{i-1}$, so that

$$\begin{aligned} C_j &= s_j \cdots s_1 C_0 \\ &= r_{i_1} \cdots r_{i_j} C_0 \text{ for } j > 0, \\ C_l &= r_{i_1} \cdots r_{i_l} C_0 \\ &= wC = D. \end{aligned}$$

Notice that $s_1 = r_1$ is the reflection in the common wall of the chambers C_0 and C_1. Next, s_j for $j > 1$ is written as

$$\begin{aligned} s_j &= r_{i_j}^{r_{i_{j-1}} \cdots r_{i_1}} \\ &= (r_{i_1} \cdots r_{i_{j-1}}) r_{i_j} (r_{i_1} \cdots r_{i_{j-1}})^{-1}. \end{aligned}$$

By Lemma 2.1.3, since r_{i_j} is a reflection in a panel, say H, of the fundamental chamber $C = C_0$, s_j is the reflection in the panel $r_{i_1} \cdots r_{i_{j-1}} H$ of the chamber $r_{i_1} \cdots r_{i_{j-1}} C_0 = C_{j-1}$. Since $s_j C_{j-1} = C_j$,

> *s_j is the reflection in the common panel of the chambers C_{j-1} and C_j.*

Summarising this procedure we obtain the following result; it will show us the right way in the labyrinth of mirrors.

Theorem 3.4.1 *Let $w = r_{i_1} \cdots r_{i_l}$ be an expression of $w \in W$ in terms of simple reflections r_i. Let C be the fundamental chamber and D a chambers in $\mathcal{C}$ such that $D = wC$. Then there exists a unique gallery $C_0, C_1, \ldots, C_l$ connecting $C = C_0$ and $D = C_l$ with the following property:*

> *$s_j = r_{i_j}^{r_{j-1}\cdots r_{i_1}}$ is the reflection in the common wall of C_{j-1} and C_j, $j = 1, \ldots, l$ and $w = s_l \cdots s_1$.*

The gallery $C_0, \ldots, C_l$ constructed in Theorem 3.4.1 will be called the *canonical w-gallery* starting at $C = C_0$.

We can reverse the above arguments and obtain also the following result.

Theorem 3.4.2 *Let $C_0, \ldots, C_l$ be a gallery connecting the fundamental chamber $C = C_0$ and a chamber $D = C_l$. Assume that the gallery does not stutter at any chamber, that is, no two consequent chambers C_i and C_{i+1} coincide. Let s_i be the reflection in the common wall of C_{i-1} and C_i, $i = 1, \ldots, l$*

Then $D = wC$ for $w = s_l \cdots s_1$,

$$C_j = s_j \cdots s_1 C_0 \ \textit{ for all } \ j = 1, \ldots, l,$$

and there exists an expression $w = r_{i_1} \cdots r_{i_l}$ of w in terms of simple reflections r_i such that, for all $j = 1, \ldots, l$,

$$s_j = r_{i_j}^{r_{j-1}\cdots r_{i_1}}.$$

3.5 Action of W on $\mathcal{C}$

In this section we shall prove arguably the most important property of the Coxeter complex.

Theorem 3.5.1 *The group W is simply transitive on $\mathcal{C}$, i.e. for any two chambers C and D in $\mathcal{C}$ there exists a unique element $w \in W$ such that $D = wC$.*

Paths. We shall call a sequence of points $\gamma_0, \ldots, \gamma_l$ a *path* if

- the consecutive points γ_{i-1} and γ_i are contained in adjacent chambers C_{i-1} and C_i;

- if $C_{i-1} = C_i$ then $\gamma_{i-1} = \gamma_i$;
- if $C_{i-1} \neq C_i$ and s_i is the reflection in the common panel of C_{i-1} and C_i then $\gamma_i = s_i\gamma_{i-1}$.

The number l is called the *length* of the path. Set $w = s_l \cdots s_0$. Since

$$\gamma_l = s_l \cdots s_1 \gamma_0 = w\gamma_0,$$

the sequence of chambers $C_0, C_1, \ldots, C_l$ is the canonical w-gallery, and, by Theorem 3.4.2, w can be expressed as a product of l simple reflections. So we have the following useful lemma.

Lemma 3.5.2 *Given a path $\gamma_0, \gamma_1, \ldots, \gamma_l$, there exists $w \in W$ such that $\gamma_l = w\gamma_0$ and $l(w) \leqslant l$.*

Notice the important property of paths: since we know that the union of two distinct adjacent closed chambers is convex (Lemma 3.1.2), the wall H_{s_i} is the only wall intersecting the segment $[\gamma_{i-1}\gamma_i]$. Therefore the following lemma holds.

Lemma 3.5.3 *If $\gamma_0, \ldots, \gamma_l$ is a path connecting the points γ_0 and γ_l lying on the opposite sides of the wall H. Then the path intersects H in the sense that, for some two consecutive points γ_{i-1} and γ_i, the wall H intersects the segment $[\gamma_{i-1}, \gamma_i]$ and*

- *the common panel of the chambers C_{i-1} and C_i containing γ_{i-1} and γ_i, respectively, belongs to H;*
- *γ_{i-1} and γ_i are symmetric in H.*

Paths and foldings. As often happens in the theory of reflection groups, an important technical result we wish to state now can be best justified by referring to a picture (Figure 3.6).

Lemma 3.5.4 *Assume that the starting point $\alpha = \gamma_0$ and the end point $\omega = \gamma_l$ of a path $\gamma_0, \ldots, \gamma_l$ lie on one side of a wall H. If the wall H intersects the path, that is, one of the points γ_i lies on the opposite side of H from α, then the path can be replaced by a shorter path with the same starting and end points, and such that it does not intersect the wall H.*

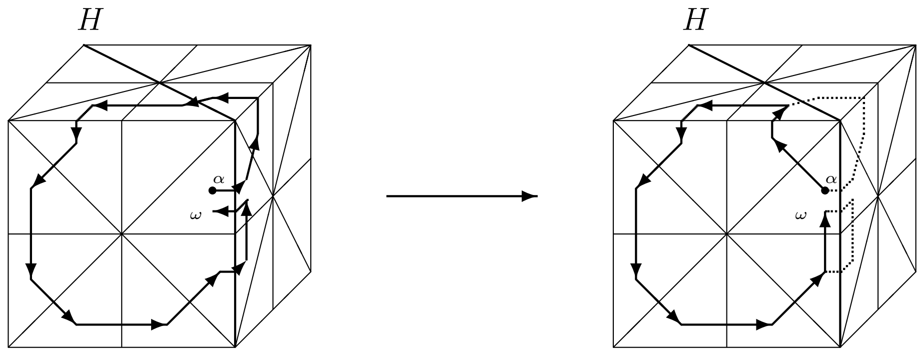


Figure 3.6: *For the proof of Lemma 3.5.4: the folding in a wall which intersects a path converts the path to a shorter one.*

Proof. See the quite self-explanatory Figure 3.6. A rigorous proof follows. However, it can be skipped in the first reading.

Assume that the path intersects the wall H at the segment $[\gamma_{p_1-1}\gamma_{p_1}]$. Then, in view of Lemma 3.5.3, the path should intersect the wall at least once more, say at the segments

$$[\gamma_{p_2-1}\gamma_{p_2}], \ldots, [\gamma_{p_k-1}\gamma_{p_k}].$$

Let $C_0, \ldots, C_l$ be the gallery corresponding to our path, so that $\gamma_i \in C_i$. Take the folding f in H onto the half space containing α and β and consider the path $f(\gamma_0), f(\gamma_1), \ldots, f(\gamma_l)$ and the gallery $f(C_0), f(C_1), \ldots, f(C_l)$. In this new gallery and new path we have repeated chambers, namely.

$$f(C_{p_1-1}) = f(C_{p_1}), \ldots, f(C_{p_k-1}) = f(C_{p_k})$$

and points

$$f(\gamma_{p_1-1}) = f(\gamma_{p_1}), \ldots, f(\gamma_{p_k-1}) = f(\gamma_{p_k})$$

After deleting the duplicate chambers and points and changing the numeration we obtain a *shorter* gallery $C_0', C_1', \ldots, C_m'$ and a path $\gamma_0', \gamma_1', \ldots, \gamma_m'$ such that $\gamma_0' = \gamma_0$, $\gamma_m' = \gamma_l$ and for all $i = 1, \ldots, m$,

- $\gamma_i' \in C_i'$;
- C_{i-1}' and C_i are adjacent;
- if s_i' is the reflection in the common wall of C_{i-1}' and C_i' then $\gamma_i' = s_i \gamma_{i-1}'$.

But then $= \gamma_0', \ldots, \gamma_m'$ is a shorter path connecting α and ω. □

Simple transitivity of W on $\mathcal{C}$: Proof of Theorem 3.5.1. In view of Corollary 3.2.2 we need to prove only the uniqueness of w. If $D = w_1C$ and $D = w_2C$ for two elements $w_1, w_2 \in W$ and $w_1 \neq w_2$, then $w_2^{-1}w_1C = C$. Denote $w = w_2^{-1}w_1$; we wish to prove $w = 1$. Assume, by way of contradiction, that $w \neq 1$. Of all expressions of w in terms of the generators $r_1, \ldots, r_n$ we take a shortest, $w = r_{i_1} \cdots r_{i_l}$, where $l = l(w)$ is the length of w. Since $w \neq 1$, $l \neq 0$. Now of all $w \in W$ with the property that $wC \neq C$ choose the one with smallest length l.

We can assume without loss of generality that C is the fundamental chamber. Let now $C_0, C_1, \ldots, C_l$ be the canonical w-gallery connecting C with C.

The vectors from the open cone C obviously span the vector space V, so the non-trivial linear transformation w cannot fix them all. Take $\gamma \in C$ such that $w\gamma \neq \gamma$ and consider the sequence of points γ_i, $i = 0, 1, \ldots, l$ defined by $\gamma_0 = \gamma$ and $\gamma_i = s_i\gamma_{i+1}$ for $i > 0$. Then $\gamma_i \in C_i$. The sequence $\gamma_0, \gamma_1, \ldots, \gamma_l$ is a path and links the end points $\gamma_0 = \gamma$ and $\gamma_l = w\gamma$. Now consider the wall $H = H_{s_1}$. Since γ_0 and γ_l both lie in C, they lie on the same side of H. But the point $\gamma_1 = s_1\gamma_0$ lies on the opposite side of H from γ. Hence, by Lemma 3.5.4, there is a shorter path connecting γ and $w\gamma$ and, by Lemma 3.5.2, an element $w' \in W$ with $w'\alpha = \omega$ and smaller length $l(w') < l$ than that of w. This contradiction completes the proof of the theorem. □

Since we have a one-to-one correspondence between positive systems, simple systems and fundamental chambers, we arrive at the following result.

Theorem 3.5.5 *The group W acts simply transitively on the set of all positive (simple) systems in Φ.*

Another important result is the following observation: for every root $\alpha \in \Phi$ the mirror H_α bounds one of the chambers in in $\mathcal{C}$. Since every chamber corresponds to some simple system in Φ and all simple systems are conjugate by Theorem 3.5.5, we come to

Theorem 3.5.6 *Let Φ be a root system, Π a simple system in Φ and W the reflection group of Φ. Every root $\alpha \in \Phi$ is conjugate, under the action of W, to a root in Π.*

Exercises

3.5.1 Prove, for involutions $r_1, \ldots, r_l$ in a group G, the identity

$$r_1 \cdots r_l = r_l^{r_{l-1} \cdots r_1} \cdot r_{l-1}^{r_{l-2} \cdots r_1} \cdots r_2^{r_1} \cdot r_1.$$

Hint: use induction on l.

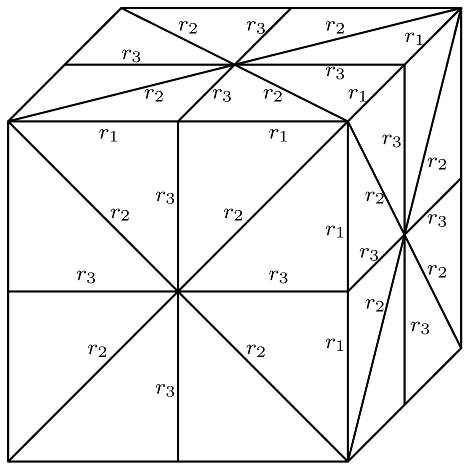


Figure 3.7: *Labelling of panels in the Coxeter complex BC_3.*

3.6 Labelling of the Coxeter complex

We shall use the simple transitivity of the action of the reflection group W on its Coxeter complex $\mathcal{C}$ to label each panel of the Coxeter complex $\mathcal{C}$ with one of the simple reflections $r_1, \ldots, r_n$; the procedure for labelling is as follows.

First we label the panels of the fundamental chamber C by the corresponding simple reflections. If D is a chamber in $\mathcal{C}$, then there is unique element $w \in W$ which sends C to $D = wC$. If Q is a panel of D, we assign to the panel Q of D the same label as that of the panel $P = w^{-1}Q$ of C.

However, we need to take care of consistency of labelling: the panel Q belongs to two adjacent chambers D and D'. If we label the panels of D' by the same rule, will the label assigned to Q be the same? Let r be the simple reflection in the panel P and $C' = rC$ the chamber adjacent to C and sharing the panel P with C. Since the action of W on $\mathcal{C}$ preserves adjacency of chambers, $D' = wC' = wrC$. Hence wr is a unique element of W which send C to D', and we assign to the panel Q the label of the panel $(wr)^{-1}Q$ of C. But $rP = P$, hence $(wr)^{-1}Qrw^{-1}Q = rP = P$, and Q gets the same label as before.

If a common panel of two chambers D and E is labelled r_i, we shall say that D and E are *r_i-adjacent*. This includes the case $D = E$, so that every chamber is r_i-adjacent to itself.

The following observation is immediate.

Proposition 3.6.1 *The action of W preserves the labelling of panels in* *the Coxeter complex $\mathcal{C}$.*

Moreover, we can now start to develope a vocabulary for translation of the geometric properties of the Coxeter complex $\mathcal{C}$ into the language of the corresponding reflection group W.

Theorem 3.6.2 *Let C be a fundamental chamber in the Coxeter complex $\mathcal{C}$ of a reflection group W. The map*

$$w \mapsto wC$$

is a one-to-one correspondence between the elements in W and chambers in $\mathcal{C}$. Two distinct chambers C and C' are r_i-adjacent if and only if the corresponding elements w and w' are related as $w' = wr_i$.

Now the description of canonical galleries given in Theorems 3.4.1 and 3.4.2 can be put in a much more convenient form.

Let $\Gamma = \{ C_0, \ldots, C_l \}$ be a gallery and let r_{i_k} the label of the common panel of the consequent chambers C_{k-1} and C_k, $k = 1, \ldots, l$. Then we say that Γ has *type* $r_{i_1}, \ldots, r_{i_l}$.

Theorem 3.6.3 *Let $\Gamma = \{ C_0, \ldots, C_l \}$ be a gallery of type $r_{i_1}, \ldots, r_{i_l}$ connecting the fundamental chamber $C = C_0$ and a chamber $D = C_l$. Set*

$$\hat{r}_{i_k} = \begin{cases} r_{i_k} & \text{if } C_{k-1} \neq C_k \\ 1 & \text{if } C_{k-1} = C_k. \end{cases}$$

Then

$$D = \hat{r}_{i_1} \cdots \hat{r}_{i_l} C.$$

For all $k = 1, \ldots, l$, the element of W corresponding to the chamber C_k is $\hat{r}_{i_1} \cdots \hat{r}_{i_k}$. In particular, if the galery Γ does not stutter, then we have, for all k, $\hat{r}_{i_k} = r_{i_k}$ and Γ is a canonical gallery for the word $w = r_{i_1} \cdots r_{i_l}$.

Proof. The proof is obvious. □

3.7 Isotropy groups

We remain in the standard setting of our study: Φ is a root system in $\mathbb{R}^n$, Σ is the corresponding mirror system and W is the reflection group.

If α is a vector in $\mathbb{R}^n$, its *isotropy group* or *stabiliser*, or *centraliser* (all these terms are used in the literature) $C_W(\alpha)$ is the group

$$C_W(\alpha) = \{ w \in W \mid w\alpha = \alpha \};$$

if $X \subseteq \mathbb{R}^n$ is a set of vectors, then its *isotropy group* or *pointwise centraliser* in W is the group

$$C_W(X) = \{\, w \in W \mid w\alpha = \alpha \text{ for all } \alpha \in X \,\}.$$

Theorem 3.7.1 *In this notation,*

(1) *The isotropy group $C_W(X)$ of a set $X \subset \mathbb{R}^n$ is generated by those reflections in W which it contains. In other words, $C_W(X)$ is generated by reflections s_H, $H \in \Sigma$, whose mirrors contain the set X.*

(2) *If X belongs to the closure $\overline{C}$ of the fundamental chamber then the isotropy group $C_W(X)$ is generated by the simple reflections it contains.*

Proof. Consider first the case when $X = \{\,\alpha\,\}$ consists of one vector. Write $W' = C_W(\alpha)$. If the vector α does not belong to any mirror in Σ then it lies in one of the open chambers in $\mathcal{C}$, say D, and $wD = D$ for any $w \in W'$. It follows from the simple transitivity of W on the Coxeter complex $\mathcal{C}$ (Theorem 3.5.1) that $W' = 1$, and the theorem is true since W' contains no reflections.

Now denote by Σ' the set of all mirrors in Σ which contain α. Obviously, Σ' is a closed mirror system and is invariant under the action of W'.

Consider the set $\mathcal{C}'$ of all chambers D such that $\alpha \in \overline{D}$. Notice that $\mathcal{C}'$ is invariant under the action of W'.

If $D \in \mathcal{C}'$ and P a panel of D containing α then the wall H of P belongs to Σ', and the chamber D' adjacent to D via the panel P belongs to $\mathcal{C}'$. Also, if two chambers $D, D' \in \mathcal{C}'$ are adjacent in $\mathcal{C}$ and have the panel P in common, then $\overline{P} = \overline{D} \cap \overline{D'}$ contains α, and the wall H containing the panel P and its closure $\overline{P}$ belongs to Σ'.

These observations allow us to prove that any two chambers D and D' in $\mathcal{C}'$ can be connected by a gallery which belongs to $\mathcal{C}'$. Indeed, let

$$D = D_0, D_1, \ldots, D_l = D'$$

be a geodesic gallery connecting D and D'. If one of the chambers in the gallery, say D_k, does not belong to $\mathcal{C}'$, then select the minimal k with this property and look at the wall H separating D_{k-1} and D_k. The chambers D, D_{k-1}, D' lie on the same side of the wall H as the point α. But a geodesic gallery intersects each wall only once. Hence the entire gallery belongs to $\mathcal{C}'$.

Now take an arbitrary $w \in W'$ and consider a gallery $D_0, \ldots, D_l$ in $\mathcal{C}'$ connecting the chambers $D = D_0$ and $D_l = wD$. If s_i is the reflection in the common panel of the consecutive chambers D_{i-1} and D_i, $i = 1, \ldots, l$, then $D' = s_l \cdots s_1 D$. Since W acts on $\mathcal{C}$ simply transitively, $w = s_l \cdots s_1$. But, for each i, $s_i \in W'$, therefore the group W' is generated by the reflections it contains. This proves (1) in our special case. The statement (2) for $X = \{\alpha\}$ follows from the observation that if $D = C$ is the fundamental chamber then the proof of the Theorem 3.2.1 can be repeated word for word for W' and $\mathcal{C}'$ and shows that W' is generated by reflections in the walls of the fundamental chamber C, i.e. by simple reflections.

Now consider the general case. If every point in X belongs to every mirror in Σ then $C_W(X) = W$ and the theorem is trivially true. Otherwise take any α in X such that the system Σ' of mirrors containing α is strictly smaller than Σ. Then $C_W(X) \leqslant C_W(\alpha)$ and $W' = C_W(\alpha)$ is itself the reflection group of Σ'. We can use induction on the number of mirrors in Σ, and application of the inductive assumption to Σ' completes the proof.
□

Exercises

3.7.1 For the symmetry group of the cube $\Delta = [-1, 1]^3$, find the isotropy groups

(a) of a vertex of the cube,

(b) of the midpoint of an edge,

(c) of the center of a 2-dimensional face.

3.7.2 Let Φ be the root system of the finite reflection group W and $\alpha \in \Phi$. Prove that the isotropy group $C_W(\alpha)$ is generated by the reflections s_β for all roots $\beta \in \Phi$ orthogonal to α.

3.7.3 The *centraliser* $C_W(u)$ of an element $w \in W$ is the set of all elements in W which commute with u:

$$C_W(w) = \{\, v \in W \mid vu = uv \,\}.$$

Let s_α be the reflection corresponding to the root $\alpha \in \Phi$. Prove that

$$C_W(s_\alpha) = \langle s_\alpha \rangle \times \langle s_\beta \mid\ \beta \in \Phi \text{ and } \beta \text{ orthogonal to } \alpha \rangle.$$

3.7.4 Let $W = \mathrm{Sym}_n$ and $r = (12)$. Prove that

$$C_W(r) = \langle (12) \rangle \times \langle (34), (45), \ldots, (n-1, n) \rangle$$

and is isomorhic to $\mathrm{Sym}_2 \times \mathrm{Sym}_{n-2}$.

3.7.5 Let Δ be a convex polytope and assume that its group of symmetries contains a subgroup W generated by reflections. If Γ is a face of Δ, prove that the set-wise stabiliser of Γ in W

$$\mathrm{Stab}_W(\Gamma) = \{\, w \in W \mid w\Gamma = \Gamma \,\}$$

is generated by reflections.

3.8 Parabolic subgroups

Let Π be a simple system in the root system Φ and $r_1, \ldots, r_m$ the corresponding system of simple reflections. Denote $I = \{\, 1, \ldots, m, \}$. For a subset $J \subseteq I$ denote

$$W_J = \langle r_i \mid i \in J \rangle;$$

subgroups W_J are called *standard parabolic subgroups* of W. Notice $W_I = W$ and $W_\emptyset = 1$.

For each $i = 1, \ldots, m$, denote by $\overline{P}_i$ the (closed) panel of the closed fundamental chamber $\overline{C}$ corresponding to the reflection r_i, and set

$$\overline{P}_J = \bigcap_{i \in J} P_J.$$

By virtue of Theorem 3.7.1,

$$W_J = C_W(\overline{P}_J).$$

We are now in a position to obtain a very easy proof of the following beautiful properties of parabolic subgroups.

Theorem 3.8.1 *If J and K are subsets of I then*

$$W_{J \cup K} = \langle W_J, W_K \rangle$$

and

$$W_{J \cap K} = W_J \cap W_K.$$

Proof. The first equality is obvious, the second one follows from the observation that

$$W_J \cap W_K = C_W(\overline{P}_J) \cap C_W(\overline{P}_K) = C_W(\overline{P}_J \cup \overline{P}_K)$$

By Theorem 3.7.1, the latter group is generated by those simple reflections whose mirrors contain the both sets $\overline{P}_J$ and $\overline{P}_K$, that is, by reflections r_i with $i \in J \cap K$. Therefore

$$W_{J\cap K} = W_J \cap W_K.$$

□

In particular, this theorem means that

$$\{ r_1, \ldots, r_n \} \cap W_J = \{ r_i \mid i \in J \}.$$

We have an important geometric interpretation of this result.

Proposition 3.8.2 *Let D and E be the chambers corresponding to the elements u and v of a parabolic subgroup P_J. If D and E are r_j-adjacent then $j \in J$.*

Proof. Since D and E are r_j-adjacent, then, by Theorem 3.6.2, we have $ur_j = v$ and $r_j \in P_J$. Therefore $j \in J$. □

Exercises

3.8.1 Let $W = A_{n-1}$ and let us view W as the symmetric group Sym_n of the set $[n]$, so that the simple reflections in W are

$$r_1 = (12), r_2 = (23), \ldots r_{n-1} = (n-1, n).$$

Prove that the parabolic subgroup

$$P = \langle r_1, \ldots, r_{k-1}, r_{k+1}, \ldots, r_{n-1} \rangle$$

is the stabiliser in Sym_n of the set $\{ 1, \ldots, k \}$ and thus is isomorphic to $\mathrm{Sym}_k \times \mathrm{Sym}_{n-k}$.

3.9 Residues

We retain the notation of the previous sections.

Let D be a chamber in $\mathcal{C}$ and $J \subset I$. A *J-residue* of D is the set of all chambers in $\mathcal{C}$ which can be connected to D by galleries in which types of panels between consequent chambers are of type r_i for $i \in J$.

Let C be the fundamental chamber of $\mathcal{C}$ and w the element of W which canonically corresponds to D, that is, $D = wC$. Then Theorem 3.6.3 says that a chamber vC belongs to the J-residue of D if and only if $v = wr_{i_1} \cdots r_{i_t}$ with $i_1, \ldots, i_t \in J$. Now we have to recall the definition of a parabolic subgroup $P_J = \langle r_i \mid i \in J \rangle$ and conclude that the chamber vC belongs to the J-residue of $D = wC$ if and only if $v \in wP_J$. Therefore J-residues are in one-two-one correspondence with the left cosets of the parabolic subgroup P_J.

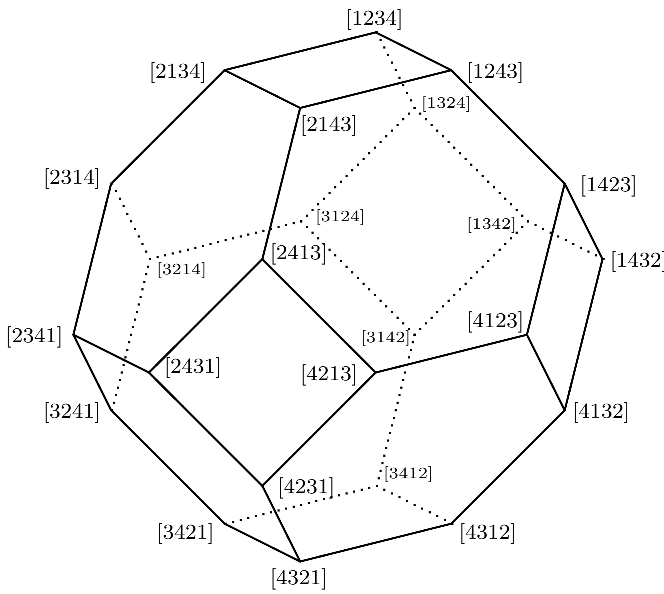


Figure 3.8: *A permutahedron for the group* $A_3 = \mathrm{Sym}_4$*. Its vertices form one orbit under the permutation action of* Sym_4 *in* $\mathbb{R}^3$ *and can be labelled by elements of* Sym_4*. Here* $[i_1 i_2 i_3 i_4]$ *denotes the permutation* $1 \mapsto i_1, \ldots, 4 \mapsto i_4$.

3.10 Generalised permutahedra

We say that a point $\alpha \in V$ is in *general position* if α does not belong to Σ.

Let now δ be any point in general position, $W \cdot \delta$ its orbit under W and Δ the convex hull of $W \cdot \delta$. We shall call Δ a *generalised permutahedron* and study it in some detail.

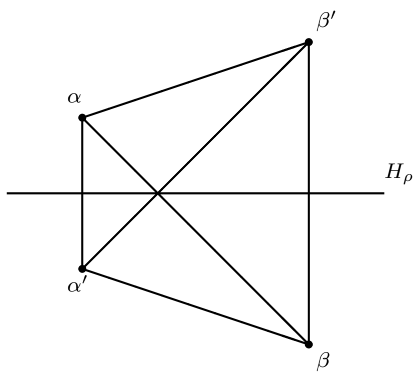
Theorem 3.10.1 *In the notation above, the following statements hold.*

(1) *Vertices of* Δ *are exactly all points in the orbit* $W \cdot \delta$ *and each chamber in* $\mathcal{C}$ *contains exactly one vertex of* Δ*.*

(2) *Every edge of* Δ *is parallel to some vector in* Φ *and intersects exactly one wall of the Coxeter complex* $\mathcal{C}$*.*

(3) *The edges emanating from the given vertex are directed along roots forming a simple system.*

(4) *If* α *is the vertex of* Δ *contained in a chamber* C *then the vertices adjacent to* α *are exactly all the mirror images* $s_i\alpha$ *of* α *in walls of* C*.*



The segment $[\alpha\beta]$ not normal to a mirror H_ρ it crosses cannot be an edge of the permutahedron Δ; indeed, if α' and β' are reflections of α and β in H_ρ then α' and β' are also vertices of Δ and $[\alpha\beta]$ belongs to the convex hull of $\alpha, \beta, \alpha', \beta'$.

Figure 3.9: *For the proof of Theorem* 3.10.1.

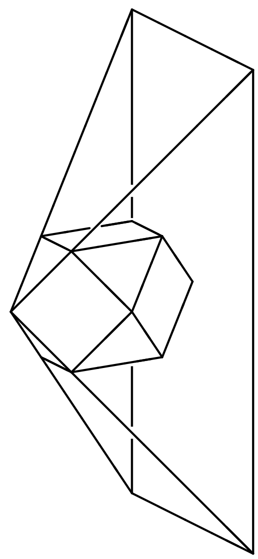
Proof. Notice, first of all, that, since all points in the orbit $W \cdot \delta$ lie at the same distance from the origin, they belong to some sphere centered at the origin. Therefore points in $W \cdot \delta$ are the vertices of the convex hull of $W \cdot \delta$. Next, because of simple transitivity of W on the Coxeter complex $\mathcal{C}$, every chamber in $\mathcal{C}$ contains exactly one vertex of Δ. This proves (1).

Let now α and β be two adjacent, i.e. connected by an edge, vertexes of Δ. Then β belongs to a chamber distinct from α and, therefore, the edge $[\alpha\beta]$ intersects some mirror H_ρ. If the edge $[\alpha\beta]$ is not perpendicular to H_ρ, we immediately have a contradiction with the following simple geometric argument (see Figure 3.9).
In Figure 3.9, the points α' and β' are symmetric to α, β, respectively, and the convex quadrangle $\alpha\alpha'\beta\beta'$ lies in a 2-dimensional plane perpendicular to the mirror H_ρ of symmetry. Therefore the segment $[\alpha\beta]$ belongs to the interior of the quadrangle and cannot be an edge of Δ.

Hence $[\alpha\beta]$ is perpendicular to H_ρ, hence $\beta - \alpha = c\rho$ for some c and the mirror H_ρ is uniquely determined, which proves (2).

Now, select a linear functional f which attains its minimum on Δ at the point α and does not vanish at roots in Φ. Let Φ^+ and Π be the positive and simple system in Φ associated with f. If $s_{\pm\rho} = s_\rho = s_{-\rho}$ is the reflection in W for the roots $\pm\rho$, then $s_{\pm\rho}\alpha$ is a vertex of Δ and $f(s_{\pm\rho}\alpha - \alpha) > 0$. But $s_{\pm\rho}\alpha - \alpha = c\rho$ for some c. After swapping notation for $+\rho$ and $-\rho$ we can assume without loss that $f(\rho) > 0$, i.e. $\rho \in \Phi^+$ and $c > 0$. Let $\beta_1, \ldots, \beta_m$ be all vertices of Δ adjacent to α. Then $\beta_i - \alpha = c_i\rho_i$ for some $\rho_i \in \Phi^+$ and $c_i > 0$.

And here comes the punchline: notice that the convex polytope Δ is contained in the convex cone Γ spanned by the edges emanating from α (Figure 3.10). Since every positive roots $\rho \in \Phi^+$ points from the vertex α to the vertex $s_\rho\alpha$ of Δ, all positive roots lie in the convex cone spanned by



One of the simple principles of linear programming which is extremely useful in the study of Coxeter groups: a convex polytope is contained in the convex polyhedral cone spanned by the edges emanating from the given vertex.

Figure 3.10: For the proof of Theorem 3.10.1.

the roots $\rho_i \in \Phi^+$ pointing from α to adjacent to α vertices β_i. But this means exactly that ρ_i form the simple system Π in Φ^+, which proves (3). Also, the fact that $\beta_i - \alpha = c\rho_i$ for $c > 0$ means that $\alpha \in V^-_{\rho_i}$. Since this holds for all simple roots, α belongs to the fundamental chamber $C = \bigcap V^-_{\rho_i}$ (Theorem 3.1.1). But, by the same theorem, C is bounded by the mirrors of simple reflections and $\beta_i = s_{\rho_i}\alpha$ is the mirror image of α in the wall H_{ρ_i} containing a panel of C. (4) is also proven. □

Exercises

3.10.1 Sketch permutahedra for the reflection groups

$$A_1 + A_1,\ A_2,\ BC_2,\ A_1 + A_1 + A_1,\ A_2 + A_1.$$

3.10.2 Label, in a way analogous to Figure 3.8, the vertices of a permutahedron for the hyperoctahedral group BC_3 (Figure 3.11) by elements of the group.

3.10.3 Let Δ be a permutahedron for a reflection group W. Prove that there is a one-to-one correspondence between faces of Δ and residues in the Coxeter complex $\mathcal{C}$ of W. Namely, the set of chambers containing vertices of a given face is a residue.

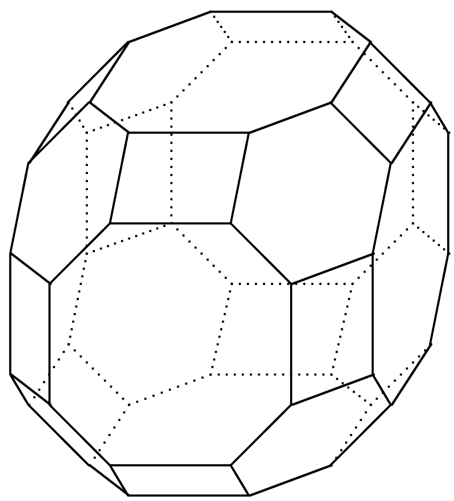


Figure 3.11: *A permutahedron for* BC_3 (*Exercise* 3.10.2).

Chapter 4

Classification

4.1 Generators and relations

Let Wbe a finite reflection group and $R = \{r_1, \ldots, r_m\}$ the set of simple reflections in W. Denote $m_{ij} = |r_i r_j|$. Notice $m_{ii} = 1$ for all i.

Theorem 4.1.1 *The group W is given by the following generators and relations:*

$$W = \langle r_1, \ldots, r_m \mid (r_i r_j)^{m_{ij}} = 1 \rangle.$$

Proof. □

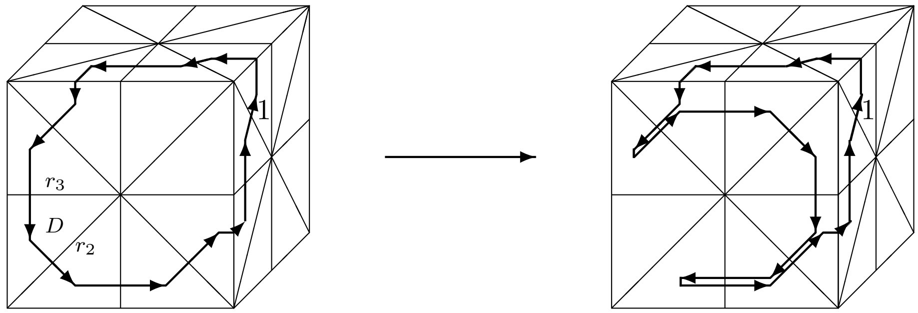
4.2 Decomposable reflection groups

Coxeter graph. By Theorem 4.1.1, a a finite reflection group W is given by the following generators and relations:

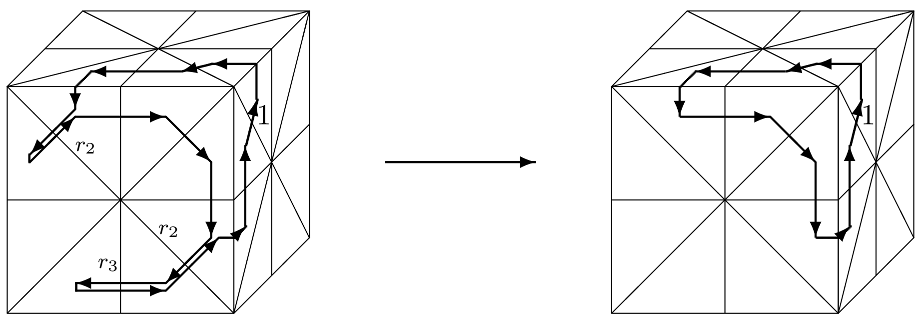
$$W = \langle r_1, \ldots, r_m \mid (r_i r_j)^{m_{ij}} = 1 \rangle$$

where $R = \{r_1, \ldots, r_m\}$ is the set of simple reflections in W and $m_{ij} = |r_i r_j|$. Notice $m_{ii} = 1$ for all i.

Now we wish to associate with W and a system of simple reflections R a graph G, called *Coxeter graph*, whose nodes are in one-to-one correspondence with the simple reflections $r_1, \ldots, r_n$ in R. If r_i and r_j are two distinct reflections, then, if $m_{ij} = |r_i r_j| > 2$, the nodes r_i and r_j are connected by an edge with mark m_{ij} on it. If $m_{ij} = 2$, that is, if r_i and r_j commute, then there is no edge connecting r_i and r_j. We say that the group W is *indecomposable* if the graph G is connected; otherwise W is said to be *decomposable*.



Removing a chamber D from a circular gallery. Here we use the relation $r_3r_2 = r_2r_3r_2r_3r_2r_3$ which is a consequence of $(r_2r_3)^4 = 1$.



Removing dead end and repeated chambers from a circular gallery. We use the relations $r_2^2 = r_3^2 = 1$.

Figure 4.1: *For the proof of Theorem* 4.1.1.

Theorem 4.2.1 *Assume that W is decomposable and let $G_1, \ldots, G_k$ be connected components of G. Let R_j be the set of reflections corresponding to nodes in the connected component G_j, $j = 1, \ldots, k$. Let W^j be the parabolic subgroup generated by the set R_j. Then*

$$W = W^1 \times \cdots \times W^k.$$

Proof. If $i \neq j$ then any two reflections $r' \in R_i$ and $r'' \in R_j$ commute, hence

- the subgroups $W^i = \langle R_i \rangle$ and $W^j = \langle R_j \rangle$ commute elementwise.

By Theorem 3.8.1, the intersection of W^j with the group generated by all W^i with $i \neq j$ is the subgroup generated by $R_j \cap \bigcup_{i \neq j} R_i = \emptyset$, that is, the identity subgroup:

- $W^j \cap \langle W^1, \dots, \widehat{W^j}, \dots, W^k \rangle = 1.$

Finally, the subgroups W^i generate W,

- $W = \langle W^1, \dots, W^k \rangle.$

But these properties of the subgroups W^i mean exactly that

$$W = W^1 \times \cdots \times W^k.$$

□

4.3 Classification of finite reflection groups

4.4 Construction of root systems

In this section we shall construct, for each Coxeter graph of type

$$A_n,\ B_n,\ C_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2$$

a root system. An immediate computation shows that all these systems are crystallographic. We do not consider the root systems of type H_3 and H_4. The interested reader may wish to consult the books [**GB**] and [**H**] which contain a detailed discussion of these non-crystallographic root systems. We notice only that the mirror system associated with the root system of type H_3 is the system of mirrors of symmetry of the regular icosahedron.

Root system A_n. Let $\epsilon_1, \dots, \epsilon_{n+1}$ be the standard basis in $\mathbb{R}^{n+1}$.

$$\begin{aligned} \Phi &= \{\, \epsilon_i - \epsilon_j \mid i, j = 1, \dots, n+1, i \neq j \,\}, \\ \Pi &= \{\, \epsilon_2 - \epsilon_1, \dots, \epsilon_{n+1} - \epsilon_n \,\}. \end{aligned}$$

Φ contains $n(n+1)$ vectors, all of whose are of equal length. Denote the simple vectors as

$$\rho_1 = \epsilon_2 - \epsilon_1,\ \rho_2 = \epsilon_3 - \epsilon_2,\ \dots,\ \rho_n = \epsilon_{n+1} - \epsilon_n$$

and take the root

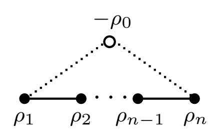
$$\begin{aligned} \rho_0 &= \epsilon_{n+1} - \epsilon_1 \\ &= \rho_1 + \rho_2 + \cdots + \rho_n. \end{aligned}$$

The root ρ_0 is called the *highest* root because it has, of all positive roots, the longest expression in terms of the simple roots. The highest root plays an exceptionally important role in many applications of the theory of root systems, for example, in the representation theory of simple Lie algebras and simple algebraic groups.

In the following diagram the black nodes form the Coxeter graph for A_n; an extra white node demonstrates the relations of the root $-\rho_0$ to the simple roots. We use the following convention: if α and β are two roots, then their nodes are not connected if $(\alpha,\beta)=0$ (and the reflections s_α and s_β commute), and the nodes are connected by an edge if

$$\frac{(\alpha,\beta)}{|\alpha||\beta|} = -\cos\frac{\pi}{m}$$

and $m \geqslant 3$. In fact, m is the order of the product $s_\alpha s_\beta$ and $m \geqslant 3$ if and only if the reflections s_α and s_β do not commute. If $m > 3$ we write the mark m on the edge.



We know that the reflection group W for our root system is the symmetric group Sym_{n+1} which acts by permuting the vectors ϵ_i.

Root system B_n, $n \geqslant 2$. Let $\epsilon_1,\dots,\epsilon_n$ be the standard basis in $\mathbb{R}^n$.

$$\begin{aligned}\Phi &= \{\pm\epsilon_i,\ \pm\epsilon_i \pm e_j \mid i,j=1,\dots,n, i<j\},\\ \Pi &= \{\epsilon_1, \epsilon_2-\epsilon_1,\dots,\epsilon_n-\epsilon_{n-1}\}.\end{aligned}$$

Φ contains $2n$ short roots $\pm\epsilon_i$ and $2n(n-1)$ long roots $\pm\epsilon_i\pm\epsilon_j$, $i<j$.

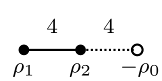
It is convenient to enumerate the simple roots as

$$\rho_1=\epsilon_1,\ \rho_2=\epsilon_2-\epsilon_1,\ \dots,\ \rho_n=\epsilon_n-\epsilon_{n-1}.$$

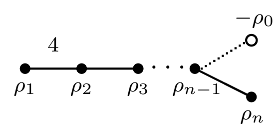
The highest root is

$$\rho_0=\epsilon_{n-1}+\epsilon_n.$$

The extended Coxeter diagrams for the system of roots B_2 and B_n with $n\geqslant 3$ look differently.



Extended Coxeter diagram for B_2



Extended Coxeter diagram for B_n, $n \geqslant 3$.

Root system C_n, $n \geqslant 2$. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis in $\mathbb{R}^n$.

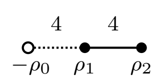
$$
\begin{aligned}
\Phi &= \{\, \pm 2\epsilon_i,\ \pm\epsilon_i \pm e_j \mid i,j = 1,\ldots,n, i<j \,\},\\
\Pi &= \{\, 2\epsilon_1, \epsilon_2 - \epsilon_1, \ldots, \epsilon_n - \epsilon_{n-1} \,\}.
\end{aligned}
$$

Φ contains $2n$ long roots $\pm 2\epsilon_i$ and $2n(n-1)$ short roots $\pm\epsilon_i \pm \epsilon_j$, $i<j$. We enumerate the simple roots as

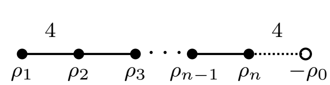
$$
\rho_1 = 2\epsilon_1,\ \rho_2 = \epsilon_2 - \epsilon_1,\ \ldots,\ \rho_n = \epsilon_n - \epsilon_{n-1}.
$$

The highest root is

$$
\rho_0 = 2\epsilon_n.
$$



Extended Coxeter diagram for C_2



Extended Coxeter diagram for C_n, $n \geqslant 3$.

Root system D_n, $n \geqslant 4$. Let $\epsilon_1, \ldots, \epsilon_n$ be the standard basis in $\mathbb{R}^n$.

$$\Phi = \{ \pm\epsilon_i \pm \epsilon_j \mid i, j = 1, 2, \ldots, n, i \neq j \};$$

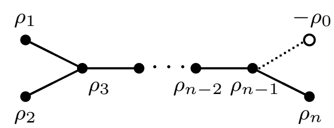
thus D_n is a subsystem of the root system C_n. All roots have the same length. The total number of roots is $2n(n-1)$.

The simple system Π is

$$\rho_1 = \epsilon_1 + \epsilon_2,\ \rho_2 = \epsilon_2 - \epsilon_1,\ \rho_3 = \epsilon_3 - \epsilon_2,\ \ldots,\ \rho_n = \epsilon_n - \epsilon_{n-1}$$

The highest root is

$$\rho_0 = \epsilon_{n-1} + \epsilon_n.$$



Root system E_8. Let $\epsilon_1, \ldots, \epsilon_8$ be the standard basis in $\mathbb{R}^8$.

$$\Phi = \left\{ \pm\epsilon_i \pm e_j \quad (i < j), \qquad \frac{1}{2}\sum_{i=1}^{8} \pm\epsilon_i \quad \text{(even number of + signs)} \right\},$$

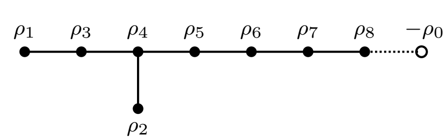
for Π take

$$\begin{aligned} \rho_1 &= \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4 - \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8), \\ \rho_2 &= \epsilon_1 + \epsilon_2, \\ \rho_i &= \epsilon_{i-1} - \epsilon_{i-2} \quad (3 \leqslant i \leqslant 8). \end{aligned}$$

All roots have the same length; the total number of roots is 240.

The highest root is

$$\rho_0 = \epsilon_7 + \epsilon_8.$$



Root system E_7. Take the root system of type E_8 in $\mathbb{R}^8$ just constructed and consider the span V of the roots $\rho_1, \ldots, \rho_7$. Let Φ be the set of 126 roots of E_8 belonging to V:

$$\pm\epsilon_i \pm \epsilon_j \quad (1 \leqslant i < j \leqslant 6),$$

$$\pm(\epsilon_7 - \epsilon_8),$$

$$\pm\frac{1}{2}\left(\epsilon_7 - \epsilon_8 + \sum_{i=1}^{6} \pm\epsilon_i\right),$$

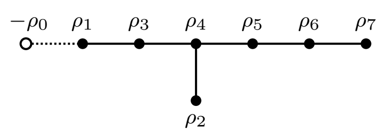
where the number of minus signs in the sum is odd.

All roots have the same length. The roots

$$\rho_1, \ldots, \rho_7$$

form a simple system, and the highest root is

$$\rho_0 = \epsilon_8 - \epsilon_7.$$



Root system E_6. Again we start with the root system of type E_8 in $\mathbb{R}^8$. Denote by V the span of the roots $\rho_1, \ldots, \rho_6$, and take for Φ the 72 roots of E_8 belonging to V:

$$\pm\epsilon_i \pm \epsilon_j \quad (1 \leqslant i < j \leqslant 5),$$

$$\pm\frac{1}{2}\left(\epsilon_8 - \epsilon_7 - \epsilon_6 + \sum_{i=1}^{5} \pm\epsilon_i\right),$$

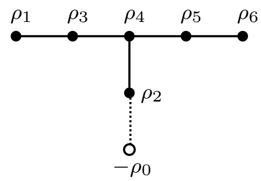
where the number of minus signs in the sum is odd.

All roots have the same length. The roots

$$\rho_1, \ldots, \rho_6$$

form a simple system, and the highest root is

$$\rho_0 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4 + \epsilon_5 - \epsilon_6 - \epsilon_7 + \epsilon_8).$$



Root system F_4. Let $\epsilon_1, \ldots, \epsilon_4$ be the standard basis in $\mathbb{R}^4$. Φ consists of 24 long roots

$$\pm\epsilon_i \pm \epsilon_j \quad (i < j)$$

and 24 short roots

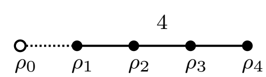
$$\pm\epsilon_i, \ \frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4).$$

For a simple system Π take

$$\rho_1 = \epsilon_2 - \epsilon_3, \ \rho_2 = \epsilon_3 - \epsilon_4, \ \rho_3 = \epsilon_4, \ \rho_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4).$$

The highest root is

$$\rho_0 = \epsilon_1 + \epsilon_2.$$



Root system G_2. Let V be the hyperplane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. Φ consists of 6 short roots

$$\pm(\epsilon_i - \epsilon_j), \quad i < j,$$

and 6 long roots

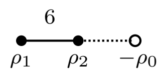
$$\pm(2\epsilon_i - \epsilon_j - \epsilon_k),$$

where i, j, k are all different. For a simple system Π take

$$\rho_1 = \epsilon_1 - \epsilon_2, \ \rho_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3.$$

The highest root is

$$\rho_0 = \epsilon_1 + \epsilon_2.$$



Exercises

For every of the root systems A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2:

4.4.1 Check the crystallographic condition.

4.4.2 Find the decomposition of the highest root with respect to the simple roots.

4.4.3 Check, by a direct computation, that the extended Coxeter diagrams are drawn correctly.

4.4.4 The sets $\Phi_{\rm long}$ and $\Phi_{\rm short}$ of all long (correspondingly, short) roots in a root system Φ are root systems on their own. Identify their types when Φ is ot type B_n, C_n or F_4.

4.5 Orders of reflection groups

In this section we shall use information about the root systems accumulated in Section 4.4 to determine the orders of the finite reflection groups

$$A_n,\ BC_n,\ D_n,\ E_6,\ E_7,\ E_8\ F_4\ G_2.$$

Notice that the group A_1, obviously, has order 2. The groups A_2, BC_2, G_2 are the dihedral groups of orders 6, 8, 12, correspondingly.

Let Φ be one of the root systems listed above and W its reflection group. To work out the order of W we need first to study the action of W on Φ.

Lemma 4.5.1 *The long (respectively, short) roots in Φ are conjugate under the action of W.*

Proof. We know from Theorem 3.5.6 that every root is conjugate to a simple root. Therefore it will be enough to prove that the simple long (respectively, short) roots are conjugate. Direct observation of Coxeter graphs shows that the nodes for any two simple roots of the same length can be connected by a sequence of edges with marks 3. Hence it will be enough to prove that if ρ_i and ρ_j are distinct simple roots so that $m_{ij} = 3$ then ρ_i and ρ_j are conjugate.

Since the system of simple roots is linearly independent, the set $\Phi' = \Phi \cap (\mathbb{Z}\rho_i + \mathbb{Z}\rho_j)$ consists of those vectors in Φ which are linear combinations of ρ_i and ρ_j. We have already checked, on several occasions, that Φ' is a root system and $\{ \rho_i, \rho_j \}$ is a simple system in Φ'. The corresponding reflection group W' is a dihedral group of order 6. One can see immediately from a simple diagram that all roots in Φ' form a single W'-orbit (check this!). Alternatively, we can argue as follows[1]: reflections in W' are in one-to-one correspondence with the 3 pairs of opposite roots in Φ'. But W' contains exactly 3 involutions, hence every involution in W' is a reflection. Every reflection in W' generates a subgroup of order 2 which is a Sylow 2-subgroup in W'. By Sylow's Theorem, all Sylow 2-subgroups are conjugate in W', hence all reflections are conjugate in W', hence all pairs of opposite roots in Φ' are conjugate, hence all roots in Φ' are conjugate in W' and, since $W' < W$, in W. □

Theorem 4.5.2 *The orders of the indecomposable reflection groups are given in the following table.*

$$
\begin{aligned}
|A_n| &= n! \\
|BC_n| &= 2^n \cdot n! \\
|D_n| &= 2^{n-1} \cdot n! \\
|E_6| &= 2^7 \cdot 3^4 \cdot 5 \\
|E_7| &= 2^{10} \cdot 3^4 \cdot 5 \cdot 7 \\
|E_8| &= 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7 \\
|F_4| &= 2^7 \cdot 3^2 \\
|G_2| &= 12
\end{aligned}
$$

Proof. In all cases the highest root ρ_0 is a long root. Since all long roots are conjugate,

$$|W| = \begin{pmatrix} \text{number} \\ \text{of long} \\ \text{roots} \end{pmatrix} \cdot |C_W(\rho_0)|.$$

[1] Which is, essentially, part of the solution to Exercise **??**

On the other hand, $C_W(\rho_0) = C_W(-\rho_0)$. One can easily check, using the formulae of Section 4.4, that $(\rho_0, \rho_i) \geqslant 0$ for all simple roots ρ_i, hence $(-\rho_0, \rho_i) \leqslant 0$ and the root $-\rho_0$ belons to the closed fundamental chamber $\overline{C}$. By Theorem 3.7.1, the isotropy group $C_W(-\rho_0)$ is generated by the simple reflections which fix the root $-\rho_0$. These simple reflections are exactly the reflections for the black nodes on the extended Coxeter graphs in Section 4.4 which are not connected by an edge to the white node $-\rho_0$. Therefore the extended Coxeter graph, with the node $-\rho_0$ and the nodes adjacent to $-\rho_0$ deleted, is the Coxeter graph for $W' = C_W(\rho_0)$, which allows us to determine the isomorphism type and the order of W'.

The rest is a case-by-case analysis.

$\boldsymbol{A_n}$. We know that $|A_1| = 2 = 2!$ and $|A_2| = 6 = 3!$. We want to prove by induction that $|A_n| = (n+1)!$. Φ contains $n(n+1)$ roots (all of them of the same length), and W' is of type A_{n-2}. By the inductive assumption, $|W'| = [(n-2)+1]! = (n-1)!$ and

$$|W| = n(n+1)\cdot(n-1)! = (n+1)!.$$

Of cource, we know that $W = Sym_{n+1}$, and there was no much need in a new proof of the fact that $|Sym_n| = n!$. But we wished to use an opportunity to show how much information about a reflection group is contained in its extended Coxeter graph.

$\boldsymbol{BC_n}$. We know that the root systems B_n and C_n have the same mirror system and reflection group. It will be more convenient for us compute with the root system C_n. It contains $2n$ long roots, and the Coxeter graph for W' is of type C_{n-1}. Thus

$$|W| = 2n\cdot 2^{n-1}(n-1)! = 2^n n!.$$

$\boldsymbol{D_n}$. All roots are long, and their number is $2n(n-1)$. The group W' has disconnected Coxeter graph with connected components of types D_{n-2} and A_1, hence $W' = W'' \times W'''$, where W'' is of type D_{n-2} has, by the inductive assumption, order $2^{n-3}(n-2)!$, and $|W'''| = 2$. Therefore

$$|W| = 2n(n-1)\cdot(2^{n-3}(n-2)!\cdot 2) = 2^{n-1}n!.$$

$\boldsymbol{E_6}$. There are 72 roots in Φ, all of them long; the isotropy group W' is of type A_5. Therefore

$$|W| = 72\cdot(5+1)! = 72\cdot 6! = 2^7\cdot 3^4\cdot 5.$$

$\boldsymbol{E_7}$. There are 126 roots in Φ, all of them long; the isotropy group W' is of type D_6 and has order $2^5\cdot 6!$. Therefore

$$|W| = 126\cdot 2^5\cdot 6! = 2^{10}\cdot 3^4\cdot 5\cdot 7.$$

$\boldsymbol{E_8}$. There are 240 roots in Φ, all of them long; the isotropy group W' is of type E_7 and has order $2^{10} \cdot 3^4 \cdot 5 \cdot 7$ (just computed). Therefore

$$|W| = 240 \cdot 2^{10} \cdot 3^4 \cdot 5 \cdot 7 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7.$$

$\boldsymbol{F_4}$. There are 24 long roots; the isotropy group W' is of type C_3 and has order $2^3 \cdot 3!$. Therefore

$$|W| = 24 \cdot 2^3 \cdot 3! = 2^7 \cdot 3^2.$$

□

Exercises

4.5.1 Prove that the roots in the root systems H_3 and H_4 form a single orbit under the action of the corresponding reflection groups.

4.5.2 Lemma 4.5.1 is not true when the root system Φ is not indecomposable. Give an example.

4.5.3 Give an example of a root system of type $A_1 \times A_1 \times A_1$ with roots of three dfferent lengths.

Bibliography

[**A**] M. A. Armstrong, **Groups and Symmetry**, Springer-Verlag, 1988.

[**Ber**] M. Berger, **Géométrie**, Nathan, 1990.

[**Bou**] N. Bourbaki, **Groupes et Algebras de Lie, Chap. 4, 5, et 6**, Hermann, Paris, 1968.

[**G**] B. Grünbaum, **Convex Polytopes**, Interscience Publishers, New York a.o., 1967.

[**GB**] L. C. Grove, C. T. Benson, **Finite Reflection Groups**, Springer-Verlag, 1984.

[**H**] J. E. Humphreys, **Reflection Groups and Coxeter Groups**, Cambridge University Press, 1990.

[**Roc**] R. T. Rockafellar, **Convex Analysis**, Princeton Univerity Press, 1970.

[**Ron**] M. Ronan, **Lectures on Buildings**, Academic Press, Boston, 1989.

[**S**] A. Schrijver, **Theory of Linear and Integer Programming**, John Wiley and Sons, Chichester a. o., 1986.

[**T**] J. Tits, *A local approach to buildings*, in **The Geometric Vein (Coxeter Festschrift)**, Springer-Verlag, New York a.o., 1981, 317–322.