GENERAL RELATIVITY & COSMOLOGY for Undergraduates

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Contents

Chapter 1

NEWTONIAN COSMOLOGY

1.1 Introduction

Many of the modern ideas in cosmology can be explained without the need to discuss General Relativity. The present chapter represents an attempt to do this based entirely on Newtonian mechanics. The equations describing the velocity (called the Friedmann equation) and acceleration of the universe are derived from Newtonian mechanics and also the cosmological constant is introduced within a Newtonian framework. The equations of state are also derived in a very simple way. Applications such as conservation laws, the age of the universe and the inflation, radiation and matter dominated epochs are discussed.

1.2 Equation of State

In what follows the equation of state for non-relativistic matter and radiation will be needed. In particular an expression for the rate of change of density, $\dot{\rho}$, will be needed in terms of the density ρ and pressure p. (The definition $\dot{x} \equiv \frac{dx}{dt}$, where t is time, is being used.) The first law of thermodynamics is

$$dU + dW = dQ \tag{1.1}$$

where U is the internal energy, W is the work and Q is the heat transfer. Ignoring any heat transfer and writing $dW = Fdr = pdV$ where F is the force, r is the distance, p is the pressure and V is the volume, then

$$dU = -pdV. \tag{1.2}$$

Assuming that ρ is a relativistic energy density means that the energy is expressed as

$$U = \rho V \tag{1.3}$$

from which it follows that

$$\dot{U} = \dot{\rho}V + \rho\dot{V} = -p\dot{V} \tag{1.4}$$

where the term on the far right hand side results from equation (1.2). Writing $V \propto r^3$ implies that $\frac{\dot{V}}{V} = 3\frac{\dot{r}}{r}$. Thus

$$\dot{\rho} = -3(\rho + p)\frac{\dot{r}}{r} \tag{1.5}$$

1.2.1 Matter

Writing the density of matter as

$$\rho = \frac{M}{\frac{4}{3}\pi r^3} \tag{1.6}$$

it follows that

$$\dot{\rho} \equiv \frac{d\rho}{dr}\dot{r} = -3\rho\frac{\dot{r}}{r} \tag{1.7}$$

so that by comparing to equation (1.5), it follows that the equation of state for matter is

$$p = 0. \tag{1.8}$$

This is the same as obtained from the ideal gas law for zero temperature. Recall that in this derivation we have not introduced any kinetic energy, so we are talking about zero temperature.

1.2.2 Radiation

The equation of state for radiation can be derived by considering radiation modes in a cavity based on analogy with a violin string [12]. For a standing wave on a string fixed at both ends

$$L = \frac{n\lambda}{2} \tag{1.9}$$

where L is the length of the string, λ is the wavelength and n is a positive integer ($n = 1, 2, 3.....$). Radiation travels at the velocity of light, so that

$$c = f\lambda = f\frac{2L}{n} \tag{1.10}$$

where f is the frequency. Thus substituting $f = \frac{n}{2L}c$ into Planck's formula $U = \hbar\omega = hf$, where h is Planck's constant, gives

$$U = \frac{nhc}{2}\frac{1}{L} \propto V^{-1/3}. \tag{1.11}$$

Using equation (1.2) the pressure becomes

$$p \equiv -\frac{dU}{dV} = \frac{1}{3}\frac{U}{V}. \tag{1.12}$$

Using $\rho = U/V$, the radiation equation of state is

$$p = \frac{1}{3}\rho. \tag{1.13}$$

It is customary to combine the equations of state into the form

$$p = \frac{\gamma}{3}\rho \tag{1.14}$$

where $\gamma \equiv 1$ for radiation and $\gamma \equiv 0$ for matter. These equations of state are needed in order to discuss the radiation and matter dominated epochs which occur in the evolution of the Universe.

1.3 Velocity and Acceleration Equations

The Friedmann equation, which specifies the speed of recession, is obtained by writing the total energy E as the sum of kinetic plus potential energy terms (and using $M = \frac{4}{3}\pi r^3\rho$)

$$E = T + V = \frac{1}{2}m\dot{r}^2 - G\frac{Mm}{r} = \frac{1}{2}mr^2(H^2 - \frac{8\pi G}{3}\rho) \tag{1.15}$$

where the Hubble constant $H \equiv \frac{\dot{r}}{r}$, m is the mass of a test particle in the potential energy field enclosed by a gas of dust of mass M, r is the distance from the center of the dust to the test particle and G is Newton's constant.

Recall that the escape velocity is just $v_{escape} = \sqrt{\frac{2GM}{r}} = \sqrt{\frac{8\pi G}{3}\rho r^2}$, so that the above equation can also be written

$$\dot{r}^2 = v_{escape}^2 - k'13 - 2 \tag{1.16}$$

with $k' \equiv -\frac{2E}{m}$. The constant k' can either be negative, zero or positive corresponding to the total energy E being positive, zero or negative. For a particle in motion near the Earth this would correspond to the particle escaping (unbound), orbiting (critical case) or returning (bound) to Earth because the speed $\dot{r}$ would be greater, equal to or smaller than the escape speed v_{escape}. Later this will be analagous to an open, flat or closed universe. Equation (1.15) is re-arranged as

$$H^2 = \frac{8\pi G}{3}\rho + \frac{2E}{mr^2}.13 - 3 \tag{1.17}$$

Defining $k \equiv -\frac{2E}{ms^2}$ and writing the distance in terms of the scale factor R and a constant length s as $r(t) \equiv R(t)s$, it follows that $\frac{\dot{r}}{r} = \frac{\dot{R}}{R}$ and $\frac{\ddot{r}}{r} = \frac{\ddot{R}}{R}$, giving the Friedmann equation

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} \tag{1.18}$$

which specifies the speed of recession. The scale factor is introduced because in General Relativity it is space itself which expands [19]. Even though this equation is derived for matter, it is also true for radiation. (In fact it is also true for vacuum, with $\Lambda \equiv 8\pi G\rho_{vac}$, where Λ is the cosmological constant and ρ_{vac} is the vacuum energy density which just replaces the ordinary density. This is discussed later.) Exactly the same equation is obtained from the general relativistic Einstein field equations [13]. According to Guth [10], k can be rescaled so that instead of being negative, zero or positive it takes on the values $-1, 0$ or $+1$. From a Newtonian point of view this corresponds to unbound, critical or bound trajectories as mentioned above. From a geometric, general relativistic point of view this corresponds to an open, flat or closed universe.

In elementary mechanics the speed v of a ball dropped from a height r is evaluated from the conservation of energy equation as $v = \sqrt{2gr}$, where g is the acceleration due to gravity. The derivation shown above is exactly analagous to such a calculation. Similarly the acceleration a of the ball is calculated as $a = g$ from Newton's equation $F = m\ddot{r}$, where F is the force and the acceleration is $\ddot{r} \equiv \frac{d^2r}{dt^2}$. The acceleration for the universe is obtained from Newton's equation

$$-G\frac{Mm}{r^2} = m\ddot{r}.13 - 5 \tag{1.19}$$

Again using $M = \frac{4}{3}\pi r^3\rho$ and $\frac{\ddot{r}}{r} = \frac{\ddot{R}}{R}$ gives the acceleration equation

$$\frac{F}{mr} \equiv \frac{\ddot{r}}{r} \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\rho. \tag{1.20}$$

However because $M = \frac{4}{3}\pi r^3\rho$ was used, it is clear that this acceleration equation holds only for matter. In our example of the falling ball instead of the acceleration being obtained from Newton's Law, it can also be obtained by taking the time derivative of the energy equation to give $a = \frac{dv}{dt} = v\frac{dv}{dr} = (\sqrt{2gr})(\sqrt{2g}\frac{1}{2\sqrt{r}}) = g$. Similarly, for the general case one can take the time derivative of equation (1.18) (valid for matter and radiation)

$$\frac{d}{dt}\dot{R}^2 = 2\dot{R}\ddot{R} = \frac{8\pi G}{3}\frac{d}{dt}(\rho R^2). \tag{1.21}$$

Upon using equation (1.5) the acceleration equation is obtained as

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3}(1+\gamma)\rho \tag{1.22}$$

which reduces to equation (1.20) for the matter equation of state ($\gamma = 0$). Exactly the same equation is obtained from the Einstein field equations [13].

1.4 Cosmological Constant

In both Newtonian and relativistic cosmology the universe is unstable to gravitational collapse. Both Newton and Einstein believed that the Universe is static. In order to obtain this Einstein introduced a *repulsive* gravitational force, called the cosmological constant, and Newton could have done exactly the same thing, had he believed the universe to be finite.

In order to obtain a possibly zero acceleration, a positive term (conventionally taken as $\frac{\Lambda}{3}$) is added to the acceleration equation (1.22) as

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{1.23}$$

which, with the proper choice of Λ can give the required zero acceleration for a static universe. Again exactly the same equation is obtained from the Einstein field equations [13]. What has been done here is entirely equivalent to just adding a repulsive gravitational force in Newton's Law. The question now is how this repulsive force enters the energy equation (1.18). Identifying the force from

$$\frac{\ddot{r}}{r} = \frac{\ddot{R}}{R} \equiv \frac{F_{repulsive}}{mr} \equiv \frac{\Lambda}{3} \tag{1.24}$$

and using

$$F_{repulsive} = \frac{\Lambda}{3} mr \equiv -\frac{dV}{dr} \tag{1.25}$$

gives the potential energy as

$$V_{repulsive} = -\frac{1}{2}\frac{\Lambda}{3} mr^2 \tag{1.26}$$

which is just a *repulsive* simple harmonic oscillator. Substituting this into the conservation of energy equation

$$E = T + V = \frac{1}{2} m\dot{r}^2 - G\frac{Mm}{r} - \frac{1}{2}\frac{\Lambda}{3} mr^2 = \frac{1}{2} mr^2 (H^2 - \frac{8\pi G}{3}\rho - \frac{\Lambda}{3}) \tag{1.27}$$

gives

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3}. \tag{1.28}$$

Equations (1.28) and (1.23) constitute the fundamental equations of motion that are used in all discussions of Friedmann models of the Universe. Exactly the same equations are obtained from the Einstein field equations [13].

Let us comment on the repulsive harmonic oscillator obtained above. Recall one of the standard problems often assigned in mechanics courses. The problem is to imagine that a hole has been drilled from one side of the Earth, through the center and to the other side. One is to show that if a ball is dropped into the hole, it will execute harmonic motion. The solution is obtained by noting that whereas gravity is an inverse square law for point masses M and m separated by a distance r as given by $F = G\frac{Mm}{r^2}$, yet if one of the masses is a continous mass distribution represented by a density then $F = G\frac{4}{3}\pi\rho mr$. The force rises linearly as the distance is increased because the amount of matter enclosed keeps increasing. Thus the gravitational force for a continuous mass distribution rises like Hooke's law and thus oscillatory solutions are encountered. This sheds light on our repulsive oscillator found above. In this case we want the gravity to be repulsive, but the cosmological constant acts just like the uniform matter distribution.

Finally authors often write the cosmological constant in terms of a vacuum energy density as $\Lambda \equiv 8\pi G\rho_{vac}$ so that the velocity and acceleration equations become

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} = \frac{8\pi G}{3}(\rho + \rho_{vac}) - \frac{k}{R^2} \tag{1.29}$$

and

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(1+\gamma)\rho + \frac{\Lambda}{3} = -\frac{4\pi G}{3}(1+\gamma)\rho + \frac{8\pi G}{3}\rho_{vac}. \tag{1.30}$$

1.4.1 Einstein Static Universe

Although we have noted that the cosmological constant provides repulsion, it is interesting to calculate its exact value for a static universe [14, 15]. The Einstein static universe requires $R = R_0 = constant$ and thus $\dot{R} = \ddot{R} = 0$. The case $\ddot{R} = 0$ will be examined first. From equation (1.23) this requires that

$$\Lambda = 4\pi G(\rho + 3p) = 4\pi G(1+\gamma)\rho. \tag{1.31}$$

If there is no cosmological constant ($\Lambda = 0$) then either $\rho = 0$ which is an empty universe, or $p = -\frac{1}{3}\rho$ which requires negative pressure. Both of these alternatives were unacceptable to Einstein and therefore he concluded that a cosmological constant was present, i.e. $\Lambda \neq 0$. From equation (1.31) this implies

$$\rho = \frac{\Lambda}{4\pi G(1+\gamma)} \tag{1.32}$$

and because ρ is positive this requires a positive Λ. Substituting equation (1.32) into equation (1.28) it follows that

$$\Lambda = \frac{3(1+\gamma)}{3+\gamma}[(\frac{\dot{R}}{R_0})^2 + \frac{k}{R_0^2}]. \tag{1.33}$$

Now imposing $\dot{R} = 0$ and assuming a matter equation of state ($\gamma = 0$) implies $\Lambda = \frac{k}{R_0^2}$. However the requirement that Λ be positive forces $k = +1$ giving

$$\Lambda = \frac{1}{R_0^2} = constant. \tag{1.34}$$

Thus the cosmological constant is not any old value but rather simply the inverse of the scale factor squared, where the scale factor has a fixed value in this static model.

Chapter 2

APPLICATIONS

2.1 Conservation laws

Just as the Maxwell equations imply the conservation of charge, so too do our velocity and acceleration equations imply conservation of energy. The energy-momentum conservation equation is derived by setting the covariant derivative of the energy momentum tensor equal to zero. The same result is achieved by taking the time derivative of equation (1.29). The result is

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} = 0. \tag{2.1}$$

This is identical to equation (1.5) illustrating the intersting connection betweeen thermodynamics and General Relativity that has been discussed recently [16]. The point is that we used thermodynamics to derive our velocity and acceleration equations and it is no surprise that the thermodynamic formula drops out again at the end. However, the velocity and acceleration equations can be obtained directly from the Einstein field equations. Thus the Einstein equations imply this thermodynamic relationship in the above equation.

The above equation can also be written as

$$\frac{d}{dt}(\rho R^3) + p\frac{dR^3}{dt} = 0 \tag{2.2}$$

and from equation (1.14), $3(\rho + p) = (3 + \gamma)\rho$, it follows that

$$\frac{d}{dt}(\rho R^{3+\gamma}) = 0. \tag{2.3}$$

Integrating this we obtain

$$\rho = \frac{c}{R^{3+\gamma}} \tag{2.4}$$

where c is a constant. This shows that the density falls as $\frac{1}{R^3}$ for matter and $\frac{1}{R^4}$ for radiation as expected.

Later we shall use these equations in a different form as follows. From equation (2.1),

$$\rho' + 3(\rho + p)\frac{1}{R} = 0 \tag{2.5}$$

where primes denote derivatives with respect to R, i.e. $x' \equiv dx/dR$. Alternatively

$$\frac{d}{dR}(\rho R^3) + 3pR^2 = 0 \tag{2.6}$$

so that

$$\frac{1}{R^{3+\gamma}}\frac{d}{dR}(\rho R^{3+\gamma}) = 0 \tag{2.7}$$

which is consistent with equation (2.4)

2.2 Age of the Universe

Recent measurements made with the Hubble space telescope [17] have determined that the age of the universe is younger than globular clusters. A possible resolution to this paradox involves the cosmological constant [18]. We illustrate this as follows.

Writing equation (1.28) as

$$\dot{R}^2 = \frac{8\pi G}{3}(\rho + \rho_{vac})R^2 - k \tag{2.8}$$

the present day value of k is

$$k = \frac{8\pi G}{3}(\rho_0 + \rho_{0vac})R_0^2 - H_0^2 R_0^2 \tag{2.9}$$

with $H^2 \equiv (\frac{\dot{R}}{R})^2$. Present day values of quantities have been denoted with a subscript 0. Substituting equation (2.9) into equation (2.8) yields

$$\dot{R}^2 = \frac{8\pi G}{3}(\rho R^2 - \rho_0 R_0^2 + \rho_{vac}R^2 - \rho_{0vac}R_0^2) - H_0^2 R_0^2. \tag{2.10}$$

Integrating gives the expansion age

$$T_0 = \int_0^{R_0} \frac{dR}{\dot{R}} = \int_0^{R_0} \frac{dR}{\sqrt{\frac{8\pi G}{3}(\rho R^2 - \rho_0 R_0^2 + \rho_{vac} R^2 - \rho_{0vac} R_0^2) - H_0^2 R_0^2}}. \tag{2.11}$$

For the cosmological constant $\rho_{vac} = \rho_{0vac}$ and because $R^2 < R_0^2$ then a non zero cosmological constant will give an age larger than would have been obtained were it not present. Our aim here is simply to show that the inclusion of a cosmological constant gives an age which is larger than if no constant were present.

2.3 Inflation

In this section only a flat $k = 0$ universe will be discussed. Results for an open or closed universe can easily be obtained and are discussed in the references [13].

Currently the universe is in a matter dominated phase whereby the dominant contribution to the energy density is due to matter. However the early universe was radiation dominated and the very early universe was vacuum dominated. Setting $k = 0$, there will only be one term on the right hand side of equation (1.29) depending on what is dominating the universe. For a matter ($\gamma = 0$) or radiation ($\gamma = 1$) dominated universe the right hand side will be of the form $\frac{1}{R^{3+\gamma}}$ (ignoring vacuum energy), whereas for a vacuum dominated universe the right hand side will be a constant. The solution to the Friedmann equation for a radiation dominated universe will thus be $R \propto t^{\frac{1}{2}}$, while for the matter dominated case it will be $R \propto t^{\frac{2}{3}}$. One can see that these results give negative acceleration, corresponding to a decelerating expanding universe.

Inflation [19] occurs when the vacuum energy contribution dominates the ordinary density and curvature terms in equation (1.29). Assuming these are negligible and substituting $\Lambda = constant$, results in $R \propto exp(t)$. The acceleration is positive, corresponding to an accelerating expanding universe called an inflationary universe.

2.4 Quantum Cosmology

2.4.1 Derivation of the Schrödinger equation

The Wheeler-DeWitt equation will be derived in analogy with the 1 dimensional Schrödinger equation, which we derive herein for completeness. The Lagrangian L for a single particle moving in a potential V is

$$L = T - V \tag{2.12}$$

where $T = \frac{1}{2}m\dot{x}^2$ is the kinetic energy, V is the potential energy. The action is $S = \int L dt$ and varying the action according to $\delta S = 0$ results in the Euler-Lagrange equation (equation of motion)

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} = 0 \tag{2.13}$$

or just

$$\dot{P} = \frac{\partial L}{\partial x} \tag{2.14}$$

where

$$P \equiv \frac{\partial L}{\partial \dot{x}}. \tag{2.15}$$

(Note P is the momentum but p is the pressure.) The Hamiltonian $\mathcal{H}$ is defined as

$$\mathcal{H}(P, x) \equiv P\dot{x} - L(\dot{x}, x). \tag{2.16}$$

For many situations of physical interest, such as a single particle moving in a harmonic oscillator potential $V = \frac{1}{2}kx^2$, the Hamiltonian becomes

$$\mathcal{H} = T + V = \frac{P^2}{2m} + V = E \tag{2.17}$$

where E is the total energy. Quantization is achieved by the operator replacements $P \to \hat{P} = -i\frac{\partial}{\partial x}$ and $E \to \hat{E} = i\frac{\partial}{\partial t}$ where we are leaving off factors of $\hbar$ and we are considering the 1-dimensional equation only. The Schrödinger equation is obtained by writing the Hamiltonian as an operator $\hat{\mathcal{H}}$ acting on a wave function Ψ as in

$$\hat{\mathcal{H}}\Psi = \hat{E}\Psi \tag{2.18}$$

and making the above operator replacements to obtain

$$(-\frac{1}{2m}\frac{\partial^2}{\partial x^2} + V)\Psi = i\frac{\partial}{\partial t}\Psi \tag{2.19}$$

which is the usual form of the 1-dimensional Schrödinger equation written in configuration space.

2.4.2 Wheeler-DeWitt equation

The discussion of the Wheeler-DeWitt equation in the minisuperspace approximation [20, 21, 11, 22] is usually restricted to closed ($k = +1$) and empty ($\rho = 0$) universes. Atkatz [11] presented a very nice discussion for closed and empty universes. Herein we consider closed, open and flat and non-empty universes. It is important to consider the possible presence of matter and radiation as they might otherwise change the conclusions. Thus presented below is a derivation of the Wheeler-DeWitt equation in the minisuperspace approximation which also includes matter and radiation and arbitrary values of k.

The Lagrangian is

$$L = -\kappa R^3 [(\frac{\dot{R}}{R})^2 - \frac{k}{R^2} + \frac{8\pi G}{3}(\rho + \rho_{vac})] \tag{2.20}$$

with $\kappa \equiv \frac{3\pi}{4G}$. The momentum conjugate to R is

$$P \equiv \frac{\partial L}{\partial \dot{R}} = -\kappa 2R\dot{R}. \tag{2.21}$$

Substituting L and P into the Euler-Lagrange equation, $\dot{P} - \frac{\partial L}{\partial R} = 0$, equation (1.29) is recovered. (Note the calculation of $\frac{\partial L}{\partial R}$ is simplified by using the conservation equation (2.5) with equation (1.14), namely $\rho' + \rho'_{vac} = -(3+\gamma)\rho/R$). The Hamiltonian $\mathcal{H} \equiv P\dot{R} - L$ is

$$\mathcal{H}(\dot{R}, R) = -\kappa R^3 [(\frac{\dot{R}}{R})^2 + \frac{k}{R^2} - \frac{8\pi G}{3}(\rho + \rho_{vac})] \equiv 0 \tag{2.22}$$

which has been written in terms of $\dot{R}$ to show explicitly that the Hamiltonian is identically zero and is not equal to the total energy as before. (Compare equation (1.29)). In terms of the conjugate momentum

$$\mathcal{H}(P, R) = -\kappa R^3 [\frac{P^2}{4\kappa^2 R^4} + \frac{k}{R^2} - \frac{8\pi G}{3}(\rho + \rho_{vac})] = 0 \tag{2.23}$$

which, of course is also equal to zero. Making the replacement $P \rightarrow -i\frac{\partial}{\partial R}$ and imposing $\mathcal{H}\Psi = 0$ results in the Wheeler-DeWitt equation in the minisuperspace approximation for arbitrary k and with matter or radiation (ρ term) included gives

$$\{-\frac{\partial^2}{\partial R^2} + \frac{9\pi^2}{4G^2}[(kR^2 - \frac{8\pi G}{3}(\rho + \rho_{vac})R^4]\}\Psi = 0. \tag{2.24}$$

Using equation (2.4) the Wheeler-DeWitt equation becomes

$$\{-\frac{\partial^2}{\partial R^2} + \frac{9\pi^2}{4G^2}[kR^2 - \frac{\Lambda}{3}R^4 - \frac{8\pi G}{3}cR^{1-\gamma}]\}\Psi = 0. \quad (2.25)$$

This just looks like the zero energy Schrödinger equation [21] with a potential given by

$$V(R) = kR^2 - \frac{\Lambda}{3}R^4 - \frac{8\pi G}{3}cR^{1-\gamma}. \quad (2.26)$$

For the empty Universe case of no matter or radiation ($c = 0$) the potential $V(R)$ is plotted in Figure 1 for the cases $k = +1, 0, -1$ respectively corresponding to closed [21], open and flat universes. It can be seen that only the closed universe case provides a potential barrier through which tunneling can occur. This provides a clear illustration of the idea that only closed universes can arise through quantum tunneling [22]. If radiation ($\gamma = 1$ and $c \neq 0$) is included then only a negative constant will be added to the potential (because the term $R^{1-\gamma}$ will be constant for $\gamma = 1$) and these conclusions about tunneling will not change. The shapes in Figure 1 will be identical except that the whole graph will be shifted downwards by a constant with the inclusion of radiation. (For matter ($\gamma = 0$ and $c \neq 0$) a term growing like R will be included in the potential which will only be important for very small R and so the conclusions again will not be changed.) To summarize, only closed universes can arise from quantum tunneling even if matter or radiation are present.

2.5 Summary

2.6 Problems

2.1

2.7 Answers

2.1

2.8 Solutions

2.1

2.2

Chapter 3

TENSORS

3.1 Contravariant and Covariant Vectors

Let us imagine that an 'ordinary' 2-dimensional vector has components (x, y) or (x^1, x^2) (read as x superscript 2 not x squared) in a certain coordinate system and components $(\overline{x}, \overline{y})$ or $(\overline{x}^1, \overline{x}^2)$ when that coordinate system is rotated by angle θ (but with the vector remaining fixed). Then the components are related by [1]

$$\begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{3.1}$$

Notice that we are using superscipts (x^i) for the components of our ordinary vectors (instead of the usual subscripts used in freshman physics), which henceforth we are going to name *contravariant vectors*. We emphasize that these are just the ordinary vectors one comes across in freshman physics.

Expanding the matrix equation we have

$$\begin{aligned} \overline{x} &= x\cos\theta + y\sin\theta \\ \overline{y} &= -x\sin\theta + y\cos\theta \end{aligned} \tag{3.2}$$

from which it follows that

$$\frac{\partial\overline{x}}{\partial x} = \cos\theta \qquad \frac{\partial\overline{x}}{\partial y} = \sin\theta \tag{3.3}$$

$$\frac{\partial \overline{y}}{\partial x} = -\sin\theta \qquad \frac{\partial \overline{y}}{\partial y} = \cos\theta$$

so that

$$\begin{aligned} \overline{x} &= \frac{\partial \overline{x}}{\partial x}x + \frac{\partial \overline{x}}{\partial y}y \\ \overline{y} &= \frac{\partial \overline{y}}{\partial x}x + \frac{\partial \overline{y}}{\partial y}y \end{aligned} \tag{3.4}$$

which can be written compactly as

$$\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^j}x^j \tag{3.5}$$

where we will always be using the Einstein summation convention for doubly repeated indices. (i.e. $x_i y_i \equiv \sum_i x_i y_i$)

Instead of defining an ordinary (contravariant) vector as a little arrow pointing in some direction, we shall instead define it as an object whose components transform according to equation(3.5). This is just a fancy version of equation(3.1), which is another way to define a vector as what happens to the components upon rotation (instead of the definition of a vector as a little arrow). Notice that we could have written down a diferential version of (3.5) just from what we know about calculus. Using the infinitessimal dx^i (instead of x^i) it follows immediately that

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^j}dx^j \tag{3.6}$$

which is identical to (3.5) and therefore we must say that dx^i forms an ordinary or contravariant vector (or an infinitessimally tiny arrow).

While we are on the subject of calculus and infinitessimals let's think about $\frac{\partial}{\partial x^i}$ which is kind of like the 'inverse' of dx^i. From calculus if $f = f(\overline{x}, \overline{y})$ and $\overline{x} = \overline{x}(x, y)$ and $\overline{y} = \overline{y}(x, y)$ (which is what (3.3) is saying) then

$$\begin{aligned} \frac{\partial f}{\partial \overline{x}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{x}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{x}} \\ \frac{\partial f}{\partial \overline{y}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \overline{y}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \overline{y}} \end{aligned} \tag{3.7}$$

or simply

$$\frac{\partial f}{\partial \overline{x}^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial \overline{x}^i}. \tag{3.8}$$

Let's 'remove' f and just write

$$\frac{\partial}{\partial \overline{x}^i} = \frac{\partial x^j}{\partial \overline{x}^i}\frac{\partial}{\partial x^j}. \tag{3.9}$$

which we see is similar to (3.5), and so we might expect that $\partial/\partial x^i$ are the 'components' of a 'non-ordinary' vector. Notice that the index is in the denominator, so instead of writing $\partial/\partial \overline{x}^i$ let's just always write it as x_i for shorthand. Or equivalently *define*

$$x_i \equiv \frac{\partial}{\partial x^i} \tag{3.10}$$

Thus

$$\overline{x}_i = \frac{\partial x^j}{\partial \overline{x}^i} x_j. \tag{3.11}$$

So now let's define a *contravariant* vector A^μ as anything whose components transform as (compare (3.5))

$$\boxed{\overline{A}^\mu \equiv \frac{\partial \overline{x}^\mu}{\partial x^\nu} A^\nu} \tag{3.12}$$

and a *covariant* vector A_μ (often also called a *one-form*, or dual vector or covector)

$$\boxed{\overline{A}_\mu = \frac{\partial x^\nu}{\partial \overline{x}^\mu} A_\nu} \tag{3.13}$$

In calculus we have two fundamental objects dx^i and the dual vector $\partial/\partial x^i$. If we try to form the dual dual vector $\partial/\partial(\partial/\partial x^i)$ we get back dx^i [2]. A set of points in a smooth space is called a manifold and where dx^i forms a space, $\partial/\partial x^i$ forms the corresponding 'dual' space [2]. The dual of the dual space is just the original space dx^i. Contravariant and covariant vectors are the dual of each other. Other examples of dual spaces are row and column matrices $(x \quad y)$ and $\begin{pmatrix} x \\ y \end{pmatrix}$ and the kets $< a|$ and bras $|a >$ used in quantum mechanics [3].

Before proceeding let's emphasize again that our definitions of contravariant and covariant vectors in (3.13) and (3.13) are nothing more than fancy versions of (3.1).

3.2 Higher Rank Tensors

Notice that our vector components A^μ have *one* index, whereas a scalar (e.g. $t =$ time or $T =$ temperature) has *zero* indices. Thus scalars are called tensors of rank zero and vectors are called tensors of rank one. We are familiar with matrices which have two indices A_{ij}. A contravariant tensor of rank two is of the form $A^{\mu\nu}$, rank three $A^{\mu\nu\gamma}$ etc. A *mixed* tensor, e.g. A^μ_ν, is partly covariant and partly contravariant.

In order for an object to be called a tensor it must satisfy the tensor transformation rules, examples of which are (3.13) and (3.13) and

$$\boxed{\overline{T}^{\mu\nu} = \frac{\partial\overline{x}^\mu}{\partial x^\alpha}\frac{\partial\overline{x}^\nu}{\partial x^\beta}T^{\alpha\beta}.} \tag{3.14}$$

$$\overline{T}^\mu_\nu = \frac{\partial\overline{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial\overline{x}^\nu}T^\alpha_\beta. \tag{3.15}$$

$$\overline{T}^{\mu\nu}_\rho = \frac{\partial\overline{x}^\mu}{\partial x^\alpha}\frac{\partial\overline{x}^\nu}{\partial x^\beta}\frac{\partial x^\gamma}{\partial\overline{x}^\rho}T^{\alpha\beta}_\gamma. \tag{3.16}$$

Thus even though a matrix has two indices A_{ij}, it may not necessarily be a second rank tensor unless it satisfies the above tensor tranformation rules as well. However all second rank tensors can be written as matrices.

Higher rank tensors can be constructed from lower rank ones by forming what is called the *outer product* or *tensor product* [14] as follows. For instance

$$T^\alpha_\beta \equiv A^\alpha B_\beta \tag{3.17}$$

or

$$T^{\alpha\beta}_{\gamma\delta} \equiv A^\alpha_\gamma B^\beta_\delta. \tag{3.18}$$

The tensor product is often written simply as

$$T = A \otimes B \tag{3.19}$$

(do Problem 3.1) (NNN Next time discuss wedge product - easy - just introduce antisymmetry).

We can also construct lower rank tensors from higher rank ones by a process called *contraction*, which sets a covariant and contravariant index equal, and because of the Einstein summation convention equal or repeated indices are summed over. Thus contraction represents setting two indices equal and summing. For example

$$T^{\alpha\beta}_{\gamma\beta} \equiv T^{\alpha}_{\gamma} \tag{3.20}$$

Thus contraction over a pair of indices reduces the rank of a tensor by two [14].

The inner product [14] of two tensors is *defined* by forming the outer product and then contracting over a pair of indices as

$$T^{\alpha}_{\beta} \equiv A^{\alpha}_{\gamma} B^{\gamma}_{\beta}. \tag{3.21}$$

Clearly the inner product of two vectors (rank one tensors) produces a scalar (rank zero tensor) as

$$A^{\mu} B_{\mu} = constant \equiv A.B \tag{3.22}$$

and it can be shown that $A.B$ as defined here is a scalar **(do Problem 3.2)**. A scalar is a tensor of rank zero with the very special transformation law of *invariance*

$$\bar{c} = c. \tag{3.23}$$

It is easily shown, for example, that $A^{\mu} B^{\mu}$ is no good as a definition of inner product for vectors because it is *not* invariant under transformations and therefore is not a scalar.

3.3 Review of Cartesian Tensors

Let us review the scalar product that we used in freshman physics. We wrote vectors as $\mathbf{A} = A_i \hat{e}_i$ and defined the scalar product as

$$\mathbf{A}.\mathbf{B} \equiv AB \cos\theta \tag{3.24}$$

where A and B are the magnitudes of the vectors $\mathbf{A}$ and $\mathbf{B}$ and θ is the angle between them. Thus

$$\begin{aligned} \mathbf{A}.\mathbf{B} &= A_i \hat{e}_i . B_j \hat{e}_j \\ &= (\hat{e}_i . \hat{e}_j) A_i B_j \\ &\equiv g_{ij} A_i B_j \end{aligned} \tag{3.25}$$

where the metric tensor g_{ij} is defined as the dot product of the basis vectors.

A Cartesian basis is defined as one in which $g_{ij} \equiv \delta_{ij}$ (obtained from $\hat{e}_i.\hat{e}_j = |\hat{e}_i||\hat{e}_j| \cos\theta = \cos\theta = \delta_{ij}$). That is, the basis vectors are of unit length and perpendicular to each other in which case

$$\begin{aligned}\mathbf{A}.\mathbf{B} &= A_i B_i \\ &= A_x B_x + A_y B_y + ... \end{aligned} \tag{3.26}$$

where the sum (+...) extends to however many dimensions are being considered and

$$\mathbf{A}.\mathbf{A} \equiv A^2 = A_i A_i \tag{3.27}$$

which is just Pythagoras' theorem, $\mathbf{A}.\mathbf{A} \equiv A^2 = A_i A_i = A_x^2 + A_y^2 + ...$

Notice that the usual results we learned about in freshman physics, equations (3.26) and (3.27), result *entirely* from requiring $g_{ij} = \delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in matrix notation.

We could easily have defined a non-Cartesian space, for example, $g_{ij} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in which case Pythagoras' theorem would change to

$$\mathbf{A}.\mathbf{A} \equiv A^2 = A_i A_i = A_x^2 + A_y^2 + A_x A_y. \tag{3.28}$$

Thus it is the metric tensor $g_{ij} \equiv \hat{e}_i.\hat{e}_j$ given by the scalar product of the unit vectors which (almost) completely defines the vector space that we are considering. Now let's return to vectors and one-forms (i.e. contravariant and covariant vectors).

3.4 Metric Tensor

We have already seen (in Problem 3.2) that the inner product defined by $A.B \equiv A_\mu B^\mu$ transforms as a scalar. (The choice $A^\mu B^\mu$ won't do because it is not a scalar). However based on the previous section, we would expect that $A.B$ can also be written in terms of a metric tensor. The most natural way to do this is

$$\begin{aligned} A.B &\equiv A_\mu B^\mu \\ &= g_{\mu\nu} A^\nu B^\mu \end{aligned} \tag{3.29}$$

assuming $g_{\mu\nu}$ is a tensor.

In fact *defining* $A.B \equiv A_\mu B^\mu \equiv g_{\mu\nu} A^\nu B^\mu$ makes perfect sense because it also transforms as a scalar (i.e. is invariant). **(do Problem 3.3)** Thus either of the two right hand sides of (3.29) will do equally well as the definition of the scalar product, and thus we deduce that

$$\boxed{A_\mu = g_{\mu\nu} A^\nu} \tag{3.30}$$

so that the metric tensor has the effect of lowering indices. Similarly it can raise indices

$$\boxed{A^\mu = g^{\mu\nu} A_\nu} \tag{3.31}$$

How is vector A written in terms of basis vectors ? Based on our experience with Cartesian vectors let's *define* our basis vectors such that

$$\begin{aligned} A.B &\equiv A_\mu B^\mu \\ &= g_{\mu\nu} A^\nu B^\mu \\ &\equiv (e_\mu . e_\nu) A^\nu B^\mu \end{aligned} \tag{3.32}$$

which imples that vectors can be written in terms of components and basis vectors as

$$\begin{aligned} A &= A^\mu e_\mu \\ &= A_\mu e^\mu \end{aligned} \tag{3.33}$$

Thus the *basis vectors of a covariant vector (one-form) transform as contravariant vectors. Contravariant components have basis vectors that transform as one-froms* [5] (pg. 63-64).

The above results illuminate our flat (Cartesian) space results where $g_{\mu\nu} \equiv \delta_{\mu\nu}$, so that (3.31) becomes $A^\mu = A_\mu$ showing that in flat space there is no distinction between covariant and contravariant vectors. Because of this it also follows that $\mathbf{A} = A_\mu \hat{e}_\mu$ and $\mathbf{A}.\mathbf{B} = A_\mu B_\mu$ which were our flat space results.

Two more points to note are the symmetry

$$g_{\mu\nu} = g_{\nu\mu} \tag{3.34}$$

and the inverse defined by

$$g_{\mu\alpha}g^{\alpha\nu} = \delta_\mu^\nu = g_\mu^\nu \tag{3.35}$$

so that g_μ^ν *is* the Kronecker delta. This follows by getting back what we start with as in $A_\mu = g_{\mu\nu}A^\nu = g_{\mu\nu}g^{\nu\alpha}A_\alpha \equiv \delta_\mu^\alpha A_\alpha$.

3.4.1 Special Relativity

Whereas the 3-dimensional Cartesian space is completely characterized by $g_{\mu\nu} = \delta_{\mu\nu}$ or

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.36}$$

Obviously for unit matrices there is no distinction between δ_μ^ν and $\delta_{\mu\nu}$. The 4-dimensional spacetime of special relativity is specified by

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3.37}$$

If a contravariant vector is specified by

$$A^\mu = (A^0, A^i) = (A^0, \mathbf{A}) \tag{3.38}$$

it follows that the covariant vector is $A_\mu = \eta_{\mu\nu}A^\nu$ or

$$A_\mu = (A_0, A_i) = (A^0, -\mathbf{A}) \tag{3.39}$$

Note that $A_0 = A^0$.

Exercise: Prove equation (3.39) using (3.38) and (3.37).

Thus, for example, the energy momentum four vector $p^\mu = (E, \mathbf{p})$ gives $p^2 = E^2 - \mathbf{p}^2$. Of course p^2 is the invariant we identify as m^2 so that $E^2 = \mathbf{p}^2 + m^2$.

Because of equation (3.38) we must have

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial x^0}, \bigtriangledown) = (\frac{\partial}{\partial t}, \bigtriangledown) \tag{3.40}$$

implying that

$$\partial^\mu \equiv \frac{\partial}{\partial x_\mu} = (\frac{\partial}{\partial x_0}, -\bigtriangledown) = (\frac{\partial}{\partial t}, -\bigtriangledown) \tag{3.41}$$

Note that $\partial^0 = \partial_0 = \frac{\partial}{\partial t}$ (with $c \equiv 1$). We define

$$\begin{aligned}\Box^2 \equiv \partial_\mu \partial^\mu &= \partial_0 \partial^0 + \partial_i \partial^i = \partial_0 \partial_0 - \partial_i \partial_i \\ &= \frac{\partial^2}{\partial x^{0^2}} - \bigtriangledown^2 = \frac{\partial^2}{\partial t^2} - \bigtriangledown^2\end{aligned} \tag{3.42}$$

(Note that some authors [30] instead define $\Box^2 \equiv \bigtriangledown^2 - \frac{\partial^2}{\partial t^2}$).

Let us now briefly discuss the fourvelocity u^μ and proper time. We shall write out c explicitly here.

Using $dx^\mu \equiv (cdt, d\mathbf{x})$ the invariant interval is

$$ds^2 \equiv dx_\mu dx^\mu = c^2 dt^2 - d\mathbf{x}^2. \tag{3.43}$$

The proper time τ is defined via

$$ds \equiv cd\tau = \frac{cdt}{\gamma} \tag{3.44}$$

which is consistent with the time dilation effect as the proper time is the time measured in an observer's rest frame. The fourvelocity is defined as

$$u^\mu \equiv \frac{dx^\mu}{d\tau} \equiv (\gamma c, \gamma \mathbf{v}) \tag{3.45}$$

such that the fourmomentum is

$$p^\mu \equiv m u^\mu = (\frac{E}{c}, \mathbf{p}) \tag{3.46}$$

where m is the rest mass.

Exercise: Check that $(mu^\mu)^2 = m^2 c^2$. (This must be true so that $E^2 = (\mathbf{p}c)^2 + (mc^2)^2$).

3.5 Christoffel Symbols

Some good references for this section are [7, 14, 8]. In electrodynamics in flat spacetime we encounter

$$\mathbf{E} = -\vec{\bigtriangledown}\phi \tag{3.47}$$

and

$$\mathbf{B} = \vec{\bigtriangledown} \times \mathbf{A} \tag{3.48}$$

where **E** and **B** are the electric and magnetic fields and ϕ and **A** are the scalar and vectors potentials. $\vec{\nabla}$ is the gradient operator defined (in 3 dimensions) as

$$\begin{aligned}\vec{\nabla} &\equiv \hat{i}\partial/\partial x + \hat{j}\partial/\partial y + \hat{k}\partial/\partial z \\ &= \hat{e}_1\partial/\partial x_1 + \hat{e}_2\partial/\partial x_2 + \hat{e}_3\partial/\partial x_3. \end{aligned} \tag{3.49}$$

Clearly then ϕ and **A** are functions of x, y, z, i.e. $\phi = \phi(x, y, z)$ and $\mathbf{A} = \mathbf{A}(x, y, z)$. Therefore ϕ is called a scalar *field* and **A** is called a vector field. **E** and **B** are also vector fields because their values are a function of position also. (The electric field of a point charge gets smaller when you move away.) Because the left hand sides are vectors, (3.47) and (3.48) imply that the derivatives $\vec{\nabla}\phi$ and $\vec{\nabla} \times \mathbf{A}$ also transform as vectors. What about the derivative of tensors in our general curved spacetime ? Do they also transform as tensors ?

Consider a vector field $A_\mu(x^\nu)$ as a function of contravariant coordinates. Let us introduce a shorthand for the derivative as

$$A_{\mu,\nu} \equiv \frac{\partial A_\mu}{\partial x^\nu} \tag{3.50}$$

We want to know whether the derivative $A_{\mu,\nu}$ is a tensor. That is does $A_{\mu,\nu}$ transform according to $\overline{A}_{\mu,\nu} = \frac{\partial x^\alpha}{\partial \overline{x}^\mu}\frac{\partial x^\beta}{\partial \overline{x}^\nu}A_{\alpha,\beta}$. ? To find out, let's evaluate the derivative explicitly

$$\begin{aligned}\overline{A}_{\mu,\nu} \equiv \frac{\partial \overline{A}_\mu}{\partial \overline{x}^\nu} &= \frac{\partial}{\partial \overline{x}^\nu}(\frac{\partial x^\alpha}{\partial \overline{x}^\mu}A_\alpha) \\ &= \frac{\partial x^\alpha}{\partial \overline{x}^\mu}\frac{\partial A_\alpha}{\partial \overline{x}^\nu} + \frac{\partial^2 x^\alpha}{\partial \overline{x}^\nu \partial \overline{x}^\mu}A_\alpha \end{aligned} \tag{3.51}$$

but A_α is a function of x^ν not $\overline{x}^\nu$, i.e. $A_\alpha = A_\alpha(x^\nu) \neq A_\alpha(\overline{x}^\nu)$. Therefore we must insert $\frac{\partial A_\alpha}{\partial \overline{x}^\nu} = \frac{\partial A_\alpha}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial \overline{x}^\nu}$ so that

$$\begin{aligned}\overline{A}_{\mu,\nu} \equiv \frac{\partial \overline{A}_\mu}{\partial \overline{x}^\nu} &= \frac{\partial x^\alpha}{\partial \overline{x}^\mu}\frac{\partial x^\gamma}{\partial \overline{x}^\nu}\frac{\partial A_\alpha}{\partial x^\gamma} + \frac{\partial^2 x^\alpha}{\partial \overline{x}^\nu \partial \overline{x}^\mu}A_\alpha \\ &= \frac{\partial x^\alpha}{\partial \overline{x}^\mu}\frac{\partial x^\gamma}{\partial \overline{x}^\nu}A_{\alpha,\gamma} + \frac{\partial^2 x^\alpha}{\partial \overline{x}^\nu \partial \overline{x}^\mu}A_\alpha \end{aligned} \tag{3.52}$$

We see therefore that the tensor transformation law for $A_{\mu,\nu}$ is spoiled by the second term. Thus $A_{\mu,\nu}$ is not a tensor [8, 7, 14].

To see why this problem occurs we should look at the definition of the derivative [8],

$$A_{\mu,\nu} \equiv \frac{\partial A_\mu}{\partial x^\nu} = \lim_{dx\to 0} \frac{A_\mu(x+dx) - A_\mu(x)}{dx^\nu} \tag{3.53}$$

or more properly [7, 14] as $\lim_{dx^\gamma\to 0} \frac{A_\mu(x^\gamma+dx^\gamma)-A_\mu(x^\gamma)}{dx^\nu}$.

The problem however with (3.53) is that the numerator is not a vector because $A_\mu(x+dx)$ and $A_\mu(x)$ are located at different points. The differnce between two vectors is only a vector if they are located at the same point. The difference betweeen two vectors located at separate points is not a vector because the transformations laws (3.12) and (3.13) depend on position. In freshman physics when we represent two vectors **A** and **B** as little arrows, the difference $\mathbf{A} - \mathbf{B}$ is not even defined (i.e. is not a vector) if **A** and **B** are at different points. We first instruct the freshman student to slide one of the vectors to the other one and only then we can visualize the difference between them. This sliding is achieved by moving one of the vectors parallel to itself (called parallel transport), which is easy to do in flat space. Thus to compare two vectors (i.e. compute $\mathbf{A} - \mathbf{B}$) we must first put them at the same spacetime point.

Thus in order to calculate $A_\mu(x+dx) - A_\mu(x)$ we must first define what is meant by parallel transport in a general curved space. When we parallel transport a vector in flat space its components don't change when we move it around, but they *do* change in curved space. Imagine standing on the curved surface of the Earth, say in Paris, holding a giant arrow (let's call this vector **A**) vertically upward. If you walk from Paris to Moscow and keep the arrow pointed upward at all times (in other words transport the vector parallel to itself), then an astronaut viewing the arrow from a stationary position in space will notice that the arrow points in different directions in Moscow compared to Paris, even though according to you, you have parallel transported the vector and it still points vertically upward from the Earth. Thus the astronaut sees the arrow pointing in a different direction and concludes that it is not the same vector. (It can't be because it points differently; it's orientation has changed.) *Thus parallel transport produces a different vector.* Vector **A** has changed into a different vector **C**.

To fix this situation, the astronaut communicates with you by radio and views your arrow through her spacecraft window. She makes a little mark on her window to line up with your arrow in Paris. She then draws a whole series of parallel lines on her window and as you walk from Paris to Moscow she keeps instructing you to keep your arrow parallel to the lines on her window. When you get to Moscow, she is satisfied that you haven't rotated your arrow compared to the markings on her window. If a vector is parallel transported from an 'absolute' point of view (the astronaut's window), then it must *still be the same vector* **A**, except now moved to a different point (Moscow).

Let's denote δA_μ as the change produced in vector $A_\mu(x^\alpha)$ located at x^α by an infinitessimal parallel transport by a distance dx^α. We expect δA_μ to be directly proportional to dx^α.

$$\delta A_\mu \propto dx^\alpha \tag{3.54}$$

We also expect δA_μ to be directly proportional to A_μ; the bigger our arrow, the more noticeable its change will be. Thus

$$\delta A_\mu \propto A_\nu dx^\alpha \tag{3.55}$$

The only sensible constant of proportionality will have to have covariant μ and α indices and a contravariant ν index as

$$\delta A_\mu \equiv \Gamma^\nu_{\mu\alpha} A_\nu dx^\alpha \tag{3.56}$$

where $\Gamma^\nu_{\mu\alpha}$ are called *Christoffel symbols* or *coefficients of affine connection* or simply *connection coefficients* . As Narlikar [7] points out, whereas the metric tensor tells us how to define distance betweeen neighboring points, the connection coefficients tell us how to define parallelism betweeen neighboring points.

Equation (3.56) defines parallel transport. δA_μ is the change produced in vector A_μ by an infinitessimal transport by a distance dx^α to produce a new vector $C_\mu \equiv A_\mu + \delta A_\mu$. To obtain parallel transport for a contravariant vector B^μ note that a scalar defined as $A_\mu B^\mu$ cannot change under parallel transport. Thus [8]

$$\delta(A_\mu B^\mu) = 0 \tag{3.57}$$

from which it follows that **(do Problem 3.4)**

$$\delta A^\mu \equiv -\Gamma^\mu_{\nu\alpha} A^\nu dx^\alpha. \tag{3.58}$$

We shall also assume [8] symmetry under exchange of lower indices,

$$\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}. \tag{3.59}$$

(We would have a truly crazy space if this wasn't true [8]. Think about it !)

Continuing with our consideration of $A_\mu(x^\alpha)$ parallel transported an infinitessimal distance dx^α, the new vector C_μ will be

$$C_\mu = A_\mu + \delta A_\mu. \tag{3.60}$$

whereas the old vector $A_\mu(x^\alpha)$ at the new position $x^\alpha + dx^\alpha$ will be $A_\mu(x^\alpha + dx^\alpha)$. The difference betweeen them is

$$dA_\mu = A_\mu(x^\alpha + dx^\alpha) - [A_\mu(x^\alpha) + \delta A_\mu] \tag{3.61}$$

which by construction *is* a vector. Thus we are led to a new definition of derivative (which *is* a tensor [8])

$$A_{\mu;\nu} \equiv \frac{dA_\mu}{dx^\nu} = \lim_{dx \to 0} \frac{A_\mu(x + dx) - [A_\mu(x) + \delta A_\mu]}{dx^\nu} \tag{3.62}$$

Using (3.53) in (3.61) we have $dA_\mu = \frac{\partial A_\mu}{\partial x^\nu} dx^\nu - \delta A_\mu = \frac{\partial A_\mu}{\partial x^\nu} dx^\nu - \Gamma^\epsilon_{\mu\alpha} A_\epsilon dx^\alpha$ and (3.62) becomes $A_{\mu;\nu} \equiv \frac{dA_\mu}{dx^\nu} = \frac{\partial A_\mu}{\partial x^\nu} - \Gamma^\epsilon_{\mu\nu} A_\epsilon$ (because $\frac{dx^\alpha}{dx^\nu} = \delta^\alpha_\nu$) which we shall henceforth write as

$$\boxed{A_{\mu;\nu} \equiv A_{\mu,\nu} - \Gamma^\epsilon_{\mu\nu} A_\epsilon} \tag{3.63}$$

where $A_{\mu,\nu} \equiv \frac{\partial A_\mu}{\partial x^\nu}$. The derivative $A_{\mu;\nu}$ is often called the *covariant derivative* (with the word covariant not meaning the same as before) and one can easily verify that $A_{\mu;\nu}$ *is* a second rank tensor (which will be done later in Problem 3.5). From (3.58)

$$\boxed{A^\mu_{;\nu} \equiv A^\mu_{,\nu} + \Gamma^\mu_{\nu\epsilon} A^\epsilon} \tag{3.64}$$

For tensors of higher rank the results are, for example, [14, 8]

$$\boxed{A^{\mu\nu}_{;\lambda} \equiv A^{\mu\nu}_{,\lambda} + \Gamma^\mu_{\lambda\epsilon} A^{\epsilon\nu} + \Gamma^\nu_{\lambda\epsilon} A^{\mu\epsilon}}$$

(3.65)

and

$$A_{\mu\nu;\lambda} \equiv A_{\mu\nu,\lambda} - \Gamma^{\epsilon}_{\mu\lambda} A_{\epsilon\nu} - \Gamma^{\epsilon}_{\nu\lambda} A_{\mu\epsilon} \tag{3.66}$$

and

$$A^{\mu}_{\nu;\lambda} \equiv A^{\mu}_{\nu,\lambda} + \Gamma^{\mu}_{\lambda\epsilon} A^{\epsilon}_{\nu} - \Gamma^{\epsilon}_{\nu\lambda} A^{\mu}_{\epsilon} \tag{3.67}$$

and

$$A^{\mu\nu}_{\alpha\beta;\lambda} \equiv A^{\mu\nu}_{\alpha\beta,\lambda} + \Gamma^{\mu}_{\lambda\epsilon} A^{\epsilon\nu}_{\alpha\beta} + \Gamma^{\nu}_{\lambda\epsilon} A^{\mu\epsilon}_{\alpha\beta} - \Gamma^{\epsilon}_{\alpha\lambda} A^{\mu\nu}_{\epsilon\beta} - \Gamma^{\epsilon}_{\beta\lambda} A^{\mu\nu}_{\alpha\epsilon}. \tag{3.68}$$

3.6 Christoffel Symbols and Metric Tensor

We shall now derive an important formula which gives the Christoffel symbol in terms of the metric tensor and its derivatives [8, 14, 7]. The formula is

$$\boxed{\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\epsilon}(g_{\epsilon\beta,\gamma} + g_{\epsilon\gamma,\beta} - g_{\beta\gamma,\epsilon}).} \tag{3.69}$$

Another result we wish to prove is that

$$\boxed{\Gamma^{\epsilon}_{\mu\epsilon} = (\ln\sqrt{-g})_{,\mu} = \frac{1}{2}[\ln(-g)]_{,\mu}} \tag{3.70}$$

where

$$g \equiv determinant|g_{\mu\nu}|. \tag{3.71}$$

Note that $g \neq |g^{\mu\nu}|$. Let us now prove these results.

Proof of Equation (3.69). The process of covariant differentiation should never change the *length* of a vector. To ensure this means that the covariant derivative of the metric tensor should always be identically zero,

$$g_{\mu\nu;\lambda} \equiv 0. \tag{3.72}$$

Applying (3.66)

$$g_{\mu\nu;\lambda} \equiv g_{\mu\nu,\lambda} - \Gamma^{\epsilon}_{\mu\lambda} g_{\epsilon\nu} - \Gamma^{\epsilon}_{\nu\lambda} g_{\mu\epsilon} \equiv 0 \tag{3.73}$$

Thus

$$g_{\mu\nu,\lambda} = \Gamma^{\epsilon}_{\mu\lambda} g_{\epsilon\nu} + \Gamma^{\epsilon}_{\nu\lambda} g_{\mu\epsilon} \tag{3.74}$$

and permuting the $\mu\nu\lambda$ indices cyclically gives

$$g_{\lambda\mu,\nu} = \Gamma^{\epsilon}_{\lambda\nu} g_{\epsilon\mu} + \Gamma^{\epsilon}_{\mu\nu} g_{\lambda\epsilon} \tag{3.75}$$

and

$$g_{\nu\lambda,\mu} = \Gamma^{\epsilon}_{\nu\mu} g_{\epsilon\lambda} + \Gamma^{\epsilon}_{\lambda\mu} g_{\nu\epsilon} \tag{3.76}$$

Now add (3.75) and (3.76) and subtract (3.74) gives [8]

$$g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda} = 2\Gamma^{\epsilon}_{\mu\nu} g_{\lambda\epsilon} \tag{3.77}$$

because of the symmetries of (3.59) and (3.34). Multiplying (3.77) by $g^{\lambda\alpha}$ and using (3.34) and (3.35) (to give $g_{\lambda\epsilon} g^{\lambda\alpha} = g_{\epsilon\lambda} g^{\lambda\alpha} = \delta^{\alpha}_{\epsilon}$) yields

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha} (g_{\lambda\mu,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}). \tag{3.78}$$

which gives (3.69). **(do Problems 3.5 and 3.6)**.

Proof of equation (3.70) [14] (Appendix II) Using $g^{\alpha\epsilon} g_{\epsilon\beta,\alpha} = g^{\alpha\epsilon} g_{\beta\alpha,\epsilon}$ (obtained using the symmetry of the metric tensor and swapping the names of indices) and contracting over $\alpha\nu$, equation (3.69) becomes (first and last terms cancel)

$$\begin{aligned} \Gamma^{\alpha}_{\beta,\alpha} &= \frac{1}{2} g^{\alpha\epsilon} (g_{\epsilon\beta,\alpha} + g_{\epsilon\alpha,\beta} - g_{\beta\alpha,\epsilon}). \\ &= \frac{1}{2} g^{\alpha\epsilon} g_{\epsilon\alpha,\beta} \end{aligned} \tag{3.79}$$

Defining g as the determinant $|g_{\mu\nu}|$ and using (3.35) it follows that

$$\frac{\partial g}{\partial g_{\mu\nu}} = g g^{\mu\nu} \tag{3.80}$$

a result which can be easily checked. **(do Problem 3.7)** Thus (3.79) becomes

$$\begin{aligned} \Gamma^{\alpha}_{\beta,\alpha} &= \frac{1}{2g} \frac{\partial g}{\partial g_{\lambda\alpha}} \frac{\partial g_{\lambda\alpha}}{\partial x^{\beta}} \\ &= \frac{1}{2g} \frac{\partial g}{\partial x^{\beta}} \\ &= \frac{1}{2} \frac{\partial \ln g}{\partial x^{\beta}} \end{aligned} \tag{3.81}$$

which is (3.70), where in (3.70) we write $\ln(-g)$ instead of $\ln g$ because g is always negative.

3.7 Riemann Curvature Tensor

The Riemann curvature tensor is one of the most important tensors in general relativity. If it is zero then it means that the space is flat. If it is non-zero then we have a curved space. This tensor is most easily derived by considering the order of double differentiation on tensors [28, 2, 9, 7, 8]. Firstly we write in general

$$A^\mu_{,\alpha\beta} \equiv \frac{\partial^2 A^\mu}{\partial x^\alpha \partial x^\beta} \tag{3.82}$$

and also when we write $A^\mu_{;\alpha\beta}$ we again mean second derivative. Many authors instead write $A^\mu_{,\alpha\beta} \equiv A^\mu_{,\alpha,\beta}$ or $A^\mu_{;\alpha\beta} \equiv A^\mu_{;\alpha;\beta}$. We shall use either notation.

In general it turns out that even though $A^\mu_{,\alpha\beta} = A^\mu_{,\beta\alpha}$, however in general it is true that $A^\mu_{;\alpha\beta} \neq A^\mu_{;\beta\alpha}$. Let us examine this in more detail. Firstly consider the second derivative of a scalar ϕ. A scalar does not change under parallel transport therefore $\phi_{;\mu} = \phi_{,\mu}$. From (3.63) we have ($\phi_{;\mu}$ is a tensor, not a scalar)

$$\phi_{;\mu;\nu} = \phi_{,\mu;\nu} = \phi_{,\mu,\nu} - \Gamma^\epsilon_{\mu\nu}\phi_{,\epsilon} \tag{3.83}$$

but because $\Gamma^\epsilon_{\mu\nu} = \Gamma^\epsilon_{\nu\mu}$ it follows that $\phi_{;\mu\nu} = \phi_{;\nu\mu}$ meaning that the order of differentiation does not matter for a scalar. Consider now a vector. Let's differentiate equation (3.64). Note that $A^\mu_{;\nu}$ is a second rank tensor, so we use (3.67) as follows

$$\begin{aligned} A^\mu_{;\nu;\lambda} &= A^\mu_{;\nu,\lambda} + \Gamma^\mu_{\lambda\epsilon}A^\epsilon_{;\nu} - \Gamma^\epsilon_{\nu\lambda}A^\mu_{;\epsilon} \\ &= \frac{\partial}{\partial x^\lambda}(A^\mu_{;\nu}) + \Gamma^\mu_{\lambda\epsilon}A^\epsilon_{;\nu} - \Gamma^\epsilon_{\nu\lambda}A^\mu_{;\epsilon} \\ &= A^\mu_{,\nu,\lambda} + \Gamma^\mu_{\nu\epsilon,\lambda}A^\epsilon + \Gamma^\mu_{\nu\epsilon}A^\epsilon_{,\lambda} + \Gamma^\mu_{\lambda\epsilon}A^\epsilon_{;\nu} - \Gamma^\epsilon_{\nu\lambda}A^\mu_{;\epsilon} \end{aligned} \tag{3.84}$$

Now interchange the order of differentiation (just swap the ν and λ indices)

$$A^\mu_{;\lambda;\nu} = A^\mu_{,\lambda,\nu} + \Gamma^\mu_{\lambda\epsilon,\nu}A^\epsilon + \Gamma^\mu_{\lambda\epsilon}A^\epsilon_{,\nu} + \Gamma^\mu_{\nu\epsilon}A^\epsilon_{;\lambda} - \Gamma^\epsilon_{\lambda\nu}A^\mu_{;\epsilon} \tag{3.85}$$

Subtracting we have

$$\begin{aligned} A^\mu_{;\nu;\lambda} - A^\mu_{;\lambda;\nu} &= A^\epsilon(\Gamma^\mu_{\nu\epsilon,\lambda} - \Gamma^\mu_{\lambda\epsilon,\nu} + \Gamma^\mu_{\lambda\theta}\Gamma^\theta_{\nu\epsilon} - \Gamma^\mu_{\nu\theta}\Gamma^\theta_{\lambda\epsilon}) \\ &\equiv A^\epsilon R^\mu_{\lambda\epsilon\nu} \end{aligned} \tag{3.86}$$

with the famous Riemann curvature tensor defined as

$$\boxed{R^{\alpha}_{\beta\gamma\delta} \equiv -\Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\beta\delta,\gamma} + \Gamma^{\alpha}_{\epsilon\gamma}\Gamma^{\epsilon}_{\beta\delta} - \Gamma^{\alpha}_{\epsilon\delta}\Gamma^{\epsilon}_{\beta\gamma}} \tag{3.87}$$

Exercise: Check that equations (3.86) and (3.87) are consistent.

The Riemann tensor tells us everything essential about the curvature of a space. For a Cartesian spcae the Riemann tensor is zero.

The Riemann tensor has the following useful symmetry properties [9]

$$R^{\alpha}_{\beta\gamma\delta} = -R^{\alpha}_{\beta\delta\gamma} \tag{3.88}$$

$$R^{\alpha}_{\beta\gamma\delta} + R^{\alpha}_{\gamma\delta\beta} + R^{\alpha}_{\delta\beta\gamma} = 0 \tag{3.89}$$

and

$$R_{\alpha\beta\gamma\delta} = -R_{\beta\alpha\gamma\delta} \tag{3.90}$$

All other symmetry properties of the Riemann tensor may be obtained from these. For example

$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta} \tag{3.91}$$

Finally we introduce the Ricci tensor [9] by contracting on a pair of indices

$$R_{\alpha\beta} \equiv R^{\epsilon}_{\alpha\epsilon\beta} \tag{3.92}$$

which has the property

$$R_{\alpha\beta} = R_{\beta\alpha} \tag{3.93}$$

(It will turn out later that $R_{\alpha\beta} = 0$ for empty space [9]). Note that the contraction of the Riemann tensor is unique up to a sign, i.e. we could have defined $R^{\epsilon}_{\epsilon\alpha\beta}$ or $R^{\epsilon}_{\alpha\epsilon\beta}$ or $R^{\epsilon}_{\alpha\beta\epsilon}$ as the Ricci tensor and we would have the same result except that maybe a sign differnce would appear. Thus different books may have this sign difference.

However all authors agree on the definition of the Riemann scalar (obtained by contracting R^{α}_{β})

$$R \equiv R^{\alpha}_{\alpha} \equiv g^{\alpha\beta} R_{\alpha\beta} \tag{3.94}$$

Finally the Einstein tensor is defined as

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \tag{3.95}$$

After discussing the stress-energy tensor in the next chapter, we shall put all of this tensor machinery to use in our discussion of general relativity following.

3.8 Summary

3.9 Problems

3.1 If A^μ and B_ν are tensors, show that the tensor product (outer product) defined by $T^\mu_\nu \equiv A^\mu B_\nu$ is also a tensor.

3.2 Show that the inner product $A.B \equiv A^\mu B_\mu$ is *invariant* under transformations, i.e. show that it satisfies the tensor transformation law of a scalar (thus it is often called the scalar product).

3.3 Show that the inner product defined by $A.B \equiv g_{\mu\nu} A^\mu B^\nu$ is also a scalar (invariant under transformations), where $g_{\mu\nu}$ is assumed to be a tensor.

3.4 Prove equation (3.58).

3.5 Derive the transformation rule for $\Gamma^\alpha_{\beta\gamma}$. Is $\Gamma^\alpha_{\beta\gamma}$ a tensor ?

3.6 Show that $A_{\mu;\nu}$ is a second rank tensor.

3.7 Check that $\frac{\partial g}{\partial g_{\mu\nu}} = g g^{\mu\nu}$. (Equation (3.80)).

3.10 Answers

no answers; only solutions

3.11 Solutions

3.1

To prove that T^μ_ν is a tensor we must show that it satisfies the tensor transformation law $\overline{T}^\mu_\nu = \frac{\partial \overline{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \overline{x}^\nu}T^\alpha_\beta$.

Proof $\overline{T}^\mu_\nu = \overline{A}^\mu \overline{B}_\nu = \frac{\partial \overline{x}^\mu}{\partial x^\alpha}A^\alpha \frac{\partial x^\beta}{\partial \overline{x}^\nu}B_\beta$

$= \frac{\partial \overline{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \overline{x}^\nu}A^\alpha B_\beta$

$= \frac{\partial \overline{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \overline{x}^\nu}T^\alpha_\beta$

QED.

3.2

First let's recall that if $f = f(\theta, \alpha)$ and $\theta = \theta(x, y)$ and $\alpha = \alpha(x, y)$ then $\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial f}{\partial x^i}\frac{\partial x^i}{\partial \theta}$.

Now

$\overline{A}.\overline{B} = \overline{A}^\mu \overline{B}_\mu = \frac{\partial \overline{x}^\mu}{\partial x^\alpha}A^\alpha \frac{\partial x^\beta}{\partial \overline{x}^\mu}B_\beta$

$= \frac{\partial \overline{x}^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial \overline{x}^\mu}A^\alpha B_\beta$

$= \frac{\partial x^\beta}{\partial x^\alpha}A^\alpha B_\beta$ by the chain rule

$= \delta^\beta_\alpha A^\alpha B_\beta$

$= A^\alpha B_\alpha$

$= A.B$

3.3

$\overline{A}.\overline{B} \equiv \overline{g}_{\mu\nu}\overline{A}^{\mu}\overline{B}^{\nu}$

$= \frac{\partial x^{\alpha}}{\partial \overline{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \overline{x}^{\nu}} g_{\alpha\beta} \frac{\partial \overline{x}^{\mu}}{\partial x^{\gamma}} \frac{\partial \overline{x}^{\nu}}{\partial x^{\delta}} A^{\gamma} B^{\delta}$

$= \frac{\partial x^{\alpha}}{\partial \overline{x}^{\mu}} \frac{\partial x^{\beta}}{\partial \overline{x}^{\nu}} \frac{\partial \overline{x}^{\mu}}{\partial x^{\gamma}} \frac{\partial \overline{x}^{\nu}}{\partial x^{\delta}} g_{\alpha\beta} A^{\gamma} B^{\delta}$

$= \frac{\partial x^{\alpha}}{\partial x^{\gamma}} \frac{\partial x^{\beta}}{\partial x^{\delta}} g_{\alpha\beta} A^{\gamma} B^{\delta}$

$= \delta^{\alpha}_{\gamma} \delta^{\beta}_{\delta} g_{\alpha\beta} A^{\gamma} B^{\delta}$

$= g_{\alpha\beta} A^{\alpha} B^{\beta}$

$= A^{\alpha} B_{\alpha}$

$= A.B$

3.4

3.5

3.6

3.7

Chapter 4

ENERGY-MOMENTUM TENSOR

It is important to emphasize that our discussion in this chapter is based entirely on Special Relativity.

4.1 Euler-Lagrange and Hamilton's Equations

Newton's second law of motion is

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{4.1}$$

or in component form (for each component F_i)

$$F_i = \frac{dp_i}{dt} \tag{4.2}$$

where $p_i = m\dot{q}_i$ (with q_i being the generalized position coordinate) so that $\frac{dp_i}{dt} = \dot{m}\dot{q}_i + m\ddot{q}_i$. If $\dot{m} = 0$ then $F_i = m\ddot{q}_i = ma_i$. For conservative forces $\mathbf{F} = -\bigtriangledown V$ where V is the scalar potential. Rewriting Newton's law we have

$$-\frac{dV}{dq_i} = \frac{d}{dt}(m\dot{q}_i) \tag{4.3}$$

Let us define the Lagrangian $L(q_i, \dot{q}_i) \equiv T - V$ where T is the kinetic energy. In freshman physics $T = T(\dot{q}_i) = \frac{1}{2}m\dot{q}_i^2$ and $V = V(q_i)$ such as the harmonic oscillator $V(q_i) = \frac{1}{2}kq_i^2$. That is in freshman physics T is a function only of velocity $\dot{q}_i$ and V is a function only of position q_i. Thus $L(q_i, \dot{q}_i) =$

$T(\dot{q}_i) - V(q_i)$. It follows that $\frac{\partial L}{\partial q_i} = -\frac{dV}{dq_i}$ and $\frac{\partial L}{\partial \dot{q}_i} = \frac{dT}{d\dot{q}_i} = m\dot{q}_i = p_i$. Thus Newton's law is

$$F_i = \frac{dp_i}{dt}$$
$$\frac{\partial L}{\partial q_i} = \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) \tag{4.4}$$

with the canonical momentum [1] defined as

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i} \tag{4.5}$$

The second equation of (4.4) is known as the Euler-Lagrange equations of motion and serves as an alternative formulation of mechanics [1]. It is usually written

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i}) - \frac{\partial L}{\partial q_i} = 0 \tag{4.6}$$

or just

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{4.7}$$

We have obtained the Euler-Lagrange equations using simple arguments. A more rigorous derivation is based on the calculus of variations [1] as discussed in Section 7.3.

We now introduce the Hamiltonian H defined as a function of p and q as

$$H(p_i, q_i) \equiv p_i\dot{q}_i - L(q_i, \dot{q}_i) \tag{4.8}$$

For the simple case $T = \frac{1}{2}m\dot{q}_i^2$ and $V \neq V(\dot{q}_i)$ we have $p_i\frac{\partial L}{\partial \dot{q}_i} = m\dot{q}_i$ so that $T = \frac{p_i^2}{2m}$ and $p_i\dot{q}_i = \frac{p_i^2}{m}$ so that $H(p_i, q_i) = \frac{p_i^2}{2m} + V(q_i) = T + V$ which is the total energy. Hamilton's equations of motion immediately follow from (4.8) as

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \tag{4.9}$$

because $L \neq L(p_i)$ and $\frac{\partial H}{\partial q_i} = -\frac{\partial L}{\partial q_i}$ so that from (4.4)

$$-\frac{\partial H}{\partial q_i} = \dot{p}_i. \tag{4.10}$$

4.2 Classical Field Theory

Scalar fields are important in cosmology as they are thought to drive inflation. Such a field is called an inflaton, an example of which may be the Higgs boson. Thus the field ϕ considered below can be thoguht of as an inflaton, a Higgs boson or any other scalar boson.

In both special and general relativity we always seek covariant equations in which space and time are given equal status. The Euler-Lagrange equations (4.6) are clearly not covariant because special emphasis is placed on time via the $\dot{q}_i$ and $\frac{d}{dt}(\frac{\partial L}{\partial \dot{q}_i})$ terms.

Let us replace the q_i by a field $\phi \equiv \phi(x)$ where $x \equiv (t, \mathbf{x})$. The generalized coordiante q has been replaced by the field variable ϕ and the discrete index i has been replaced by a continuously varying index x. In the next section we shall show how to derive the Euler-Lagrange equations from the action defined as

$$S \equiv \int L dt \tag{4.11}$$

which again is clearly not covariant. A covariant form of the action would involve a Lagrangian density $\mathcal{L}$ via

$$S \equiv \int \mathcal{L} d^4x = \int \mathcal{L} d^3x dt \tag{4.12}$$

with $L \equiv \int \mathcal{L} d^3x$. The term $-\frac{\partial L}{\partial q_i}$ in equation (4.6) gets replaced by the covariant term $-\frac{\partial \mathcal{L}}{\partial \phi(x)}$. Any time derivative $\frac{d}{dt}$ should be replaced with $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ which contains space as well as time derivatives. Thus one can guess that the covariant generalization of the point particle Euler-Lagrange equations (4.6) is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \tag{4.13}$$

which is the covariant Euler-Lagrange equation for scalar fields. This will be derived rigorously in the next section.

In analogy with the canonical momentum in equation (4.5) we define the *covariant momentum density*

$$\Pi^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \tag{4.14}$$

so that the Euler-Lagrange equations become

$$\partial_\mu \Pi^\mu = \frac{\partial \mathcal{L}}{\partial \phi} \tag{4.15}$$

The canonical momentum is defined as

$$\Pi \equiv \Pi^0 = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} \tag{4.16}$$

The energy momentum tensor is (analagous to (4.8))

$$T_{\mu\nu} \equiv \Pi_\mu \partial_\nu \phi - g_{\mu\nu}\mathcal{L} \tag{4.17}$$

with the Hamiltonian density

$$H \equiv \int \mathcal{H} d^3x$$
$$\mathcal{H} \equiv T_{00} = \Pi\dot{\phi} - \mathcal{L} \tag{4.18}$$

4.2.1 Classical Klein-Gordon Field

In order to illustrate the foregoing theory we shall use the example of the classical, massive Klein-Gordon field defined with the Lagrangian density **(HL units ??)**

$$\mathcal{L}_{KG} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2)$$
$$= \frac{1}{2}[\dot{\phi}^2 - (\bigtriangledown\phi)^2 - m^2\phi^2] \tag{4.19}$$

The covariant momentum density is more easily evaluated by re-writing $\mathcal{L}_{KG} = \frac{1}{2}g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi - m^2\phi^2)$. Thus $\Pi^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \frac{1}{2}g^{\mu\nu}(\delta^\alpha_\mu\partial_\nu\phi + \partial_\mu\phi\delta^\alpha_\nu)$ $= \frac{1}{2}(\delta^\alpha_\mu\partial^\mu\phi + \partial^\nu\phi\delta^\alpha_\nu) = \frac{1}{2}(\partial^\alpha\phi + \partial^\alpha\phi) = \partial^\alpha\phi$. Thus for the Klein-Gordon field we have

$$\Pi^\alpha = \partial^\alpha\phi \tag{4.20}$$

giving the canonical momentum $\Pi = \Pi^0 = \partial^0\phi = \partial_0\phi = \dot{\phi}$,

$$\Pi = \dot{\phi} \tag{4.21}$$

Evaluating $\frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi$, the Euler-Lagrange equations give the field equation as $\partial_\mu\partial^\mu\phi + m^2\phi$ or

$$(\Box^2 + m^2)\phi = 0$$
$$\ddot{\phi} - \bigtriangledown^2\phi + m^2\phi = 0 \tag{4.22}$$

which is the Klein-Gordon equation for a free, massive scalar field. In momentum space $p^2 = -\Box^2$, thus

$$(p^2 - m^2)\phi = 0 \tag{4.23}$$

(Note that some authors [30] define $\Box^2 \equiv \bigtriangledown^2 - \frac{\partial^2}{\partial t^2}$ different from (3.42), so that they write the Klein-Gordon equation as $(\Box^2 - m^2)\phi = 0$ or $(p^2 + m^2)\phi = 0$.)

The energy momentum tensor is

$$\begin{aligned} T_{\mu\nu} &\equiv \Pi_\mu \partial_\nu \phi - g_{\mu\nu}\mathcal{L} \\ &= \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu}\mathcal{L} \\ &= \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu}(\partial_\alpha \phi \partial^\alpha \phi - m^2\phi^2). \end{aligned} \tag{4.24}$$

Therefore the Hamiltonian density is $\mathcal{H} \equiv T_{00} = \dot{\phi}^2 - \frac{1}{2}(\partial_\alpha \phi \partial^\alpha \phi - m^2\phi^2)$ which becomes [31]

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\bigtriangledown\phi)^2 + \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}[\Pi^2 + (\bigtriangledown\phi)^2 + m^2\phi^2] \end{aligned} \tag{4.25}$$

where we have relied upon the results of Section 3.4.1.

4.3 Principle of Least Action

derive EL eqns properly for q and ϕ (do later). Leave out for now.

4.4 Energy-Momentum Tensor for Perfect Fluid

The best references for this section are [9](Pg. 124-125), [7], and [32](Pg. 155). The book by D'Inverno [32] also has a nice discussion of the Navier-Stokes equation and its relation to the material of this section. Other references are [8](Pg. 83), [15](Pg. 330), [33](Pg. 259), [34](Pg. 38), and [2].

These references show that the energy-momentum tensor for a perfect fluid is

$$\boxed{T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p\eta^{\mu\nu}} \tag{4.26}$$

where ρ is the *energy* density and p is the pressure. We shall now work this out for several specific cases [9]. Fig. 2.5 of Narlikar's book [7] is particularly helpful.

Motionless dust represents a collection of particles at rest. Thus $u^\mu = (c, \mathbf{0})$, so that $T^{00} = \rho$. The equation of state for dust is $p = 0$ so that $T^{ii} = 0 = T^{0i} = T^{ij}$. Thus

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.27}$$

Motionless fluid representes a collection of particles all *moving* randomly (such that they exert a pressure) but the whole collection is at rest, such as a gas of particles at non-zero temperature, but confined in a motionless container. In this case $u^\mu = (c, \mathbf{0})$ again, but now $p \neq 0$. Thus again $T^{00} = \rho$ but now $T^{ii} = p$ and $T^{ij} = 0$ so that

$$T^{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{4.28}$$

Motionless radiation is characterized by the equation of state $p = \frac{1}{3}\rho$. Again the radiation is confined to a container not in motion so that $u^\mu = (\gamma c, \mathbf{0})$. (The $\frac{1}{3}$ just comes from randomizing the pressure in 3 dimensions [7].) Thus

$$\begin{aligned} T^{\mu\nu} &= \frac{4}{3}\rho u^\mu u^\nu - \frac{1}{3}\rho\eta^{\mu\nu} \\ &= \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & \frac{1}{3}\rho & 0 & 0 \\ 0 & 0 & \frac{1}{3}\rho & 0 \\ 0 & 0 & 0 & \frac{1}{3}\rho \end{pmatrix} \end{aligned} \tag{4.29}$$

Thus the general case is the motionless fluid energy-momentum tensor in equation (4.28). The special cases of motionless dust or motionless radiation are obtained with the respective substitutions of $p = 0$ or $p = \frac{1}{3}\rho$ in equation (4.28).

4.5 Continuity Equation

In classical electrodynamics the fourcurrent density is $j^\mu \equiv (c\rho, \mathbf{j})$ and the covariant conservation law is $\partial_\mu j^\mu = 0$ which results in the equation of continuity $\frac{\partial \rho}{\partial t} + \bigtriangledown . \mathbf{j} = 0$. This can *also* be obtained from the Maxwell equations by taking the divergence of Ampère's law. **(do Problems 4.1 and 4.2)** Thus the four Maxwell equations are entirely equivalent to only three Maxwell equations plus the equation of continuity.

We had a similar situation in Chapters 1 and 2 where we found that the velocity and acceleration equations imply the conservation equation. Thus the two velocity and acceleration equations are entirely equivalent to only the velocity equation plus the conservation law.

In analogy with electrodynamics the conservation law for the energy-momentum tensor is

$$T^{\mu\nu}_{;\nu} = 0 \tag{4.30}$$

In the next chapter we shall show how equation (2.1) can be derived from this.

4.6 Interacting Scalar Field

We represent the interaction of a scalar field with a scalar potential $V(\phi)$. Recall our elementary results for $L = T - V = \frac{1}{2}m\dot{q}_i^2 - V(q_i)$ for the coordinates $\dot{q}_i$. These discrete coordinates $\dot{q}_i$ have now been replaced by continuous field variables $\phi(x)$ where ϕ has replaced the generalized coordinate q and the discrete index i has been replaced by a continuous index x. Thus $V(q_i)$ naturally gets replaced with $V(\phi)$ where $\phi \equiv \phi(x)$.

Thus for an interacting scalar field we simply tack on $-V(\phi)$ to the free Klein-Gordon Lagrangian of equation (4.19) to give

$$\begin{aligned}\mathcal{L} = \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - V(\phi) \\ \equiv \mathcal{L}_O + \mathcal{L}_I\end{aligned} \tag{4.31}$$

where $\mathcal{L}_O \equiv \mathcal{L}_{KG}$ and $\mathcal{L}_I \equiv -V(\phi)$. Actually the Lagrangian of (4.31) refers to a minimally coupled scalar field as opposed to conformally coupled [21] (Pg. 276). It is important to emphasize that $V(\phi)$ does not contain derivative terms such as $\partial_\mu\phi$. Thus the covariant momentum density and canonical momentum remain the same as equations (4.20) and (4.21) for the free particle case namely $\Pi^\alpha = \partial^\alpha \phi$ and $\pi = \dot{\phi}$. Solving the Euler-Lagrange equations now gives

$$\begin{aligned}(\Box^2 + m^2)\phi + V' = 0 \\ \ddot{\phi} - \bigtriangledown^2\phi + m^2\phi + V' = 0\end{aligned} \tag{4.32}$$

with

$$V' \equiv \frac{dV}{d\phi} \tag{4.33}$$

The energy-momentum tensor is the same as for the free particle case, equation (4.24), except for the addition of $g_{\mu\nu}V(\phi)$ as in

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}[\frac{1}{2}(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2) - V(\phi)] \tag{4.34}$$

yielding the Hamiltonian density the same as for the free particle case, equation (4.25), except for the addition of $V(\phi)$ as in

$$\begin{aligned}\mathcal{H} \equiv T_{00} &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\bigtriangledown\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi) \\ &= \frac{1}{2}[\Pi^2 + (\bigtriangledown\phi)^2 + m^2\phi^2] + V(\phi).\end{aligned} \tag{4.35}$$

The purely spatial components are $T_{ii} = \partial_i\phi\partial_i\phi - g_{ii}[\frac{1}{2}(\partial_\alpha\phi\partial^\alpha\phi - m^2\phi^2) - V(\phi)]$ and with $g_{ii} = -1$ (i.e. assume Special Relativity NNN) we obtain

$$T_{ii} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\bigtriangledown\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \tag{4.36}$$

Note that even though T_{ii} has repeated indices let us *not* assume $\sum_i$ is implied in this case. That is T_{ii} refers to $T_{ii} = T_{11} = T_{22} = T_{33}$ and *not* $T_{ii} = T_{11} + T_{22} + T_{33}$. Some authors (e.g Serot and Walecka [34]) do assume the latter convention and therefore will disagree with our results by $\frac{1}{3}$.

Let us assume that the effects of the scalar field are averaged so as to behave like a perfect (motionless) fluid. In that case, comparing equation (4.28), we make the identification [13, 34]

$$\mathcal{E} \equiv \rho \equiv < T_{00} > \tag{4.37}$$

and

$$p \equiv < T_{ii} > \tag{4.38}$$

where $\mathcal{E} \equiv \rho$ is the energy density and p is the pressure. (Note that because Serot and Walecka do assume the Einstein summation convention for T_{ii}, they actually write $p \equiv \frac{1}{3} < T_{ii} >$.) Making these identifications we have

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\bigtriangledown\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi) \tag{4.39}$$

and

$$p = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\bigtriangledown\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi) \tag{4.40}$$

Let us also assume that the scalar field is massless and that $\phi = \phi(t)$ only, i.e. $\phi \neq \phi(\mathbf{x})$, so that spatial derivatives disappear. (See Pg. 276-277 of Kolb and Turner [21] and Pg. 138 of Islam [13]). Therefore we finally obtain [13, 21].

$$\boxed{\rho = \tfrac{1}{2}\dot{\phi}^2 + V(\phi)} \tag{4.41}$$

and

$$\boxed{p = \tfrac{1}{2}\dot{\phi}^2 - V(\phi)} \tag{4.42}$$

4.7 Cosmology with the Scalar Field

We have finished with our discussion of the energy-momentum tensor and therefore we should now move onto the next chapter. However, with the tools at hand (energy-momentum tensor and Friedmann equations) we can discuss the relevance of the scalar field to cosmology *without needing the formalism of General Relativity*. Therefore before proceeding to the next chapter we shall make a brief digression and discuss the evolution of the scalar field.

If one is considering cosmological evolution driven by a scalar field, one can simply substitute the above expressions for ρ and p into the Friedmann and acceleration equations (1.29) and (1.30) to obtain the time evolution of the scale factor as in

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + V(\phi)] - \frac{k}{R^2} + \frac{\Lambda}{3} \tag{4.43}$$

and

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}[\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)] + \frac{\Lambda}{3} \tag{4.44}$$

The equation for the time evolution of the scalar field is obtained either by taking the time derivative of equation (4.43) or more simply by substituting the expression for ρ and p in equations (4.41) and (4.42) into the conservation equation (2.1) to give

$$\ddot{\phi} + 3H[\dot{\phi} + \frac{(\nabla\phi)^2}{\dot{\phi}}] + m^2\phi + V' = 0. \tag{4.45}$$

Note that this is a new Klein-Gordon equation quite different to equation (4.32). The difference occurs because we have now incorporated gravity via the Friedmann and conservation equation. We shall derive this equation again in Chapter 7.

Again assuming the field is massless and ignoring spatial derivatives we have

$$\boxed{\ddot{\phi} + 3H\dot{\phi} + V' = 0} \tag{4.46}$$

Notice that this is the equation for a damped harmonic oscillator ($V = \frac{1}{2}kx^2$ and $\frac{dV}{dx} \equiv V' = kx$ with $F = -V'$) as

$$m\ddot{x} + d\dot{x} + kx = 0 \tag{4.47}$$

Kolb and Turner [21] actually also include a particle creation term due to the decay of the scalar field, which will cause reheating, and instead write

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma\dot{\phi} + V' = 0 \tag{4.48}$$

4.7.1 Alternative derivation

We can derive the equation of motion (4.46) for the scalar field in a quicker manner [29] (Pg. 73), but *this derivation only seems to work if we set* $m = 0$ *and* $\bigtriangledown\phi = 0$ *at the beginning.* (*Exercise:* find out what goes wrong if $m \neq 0$ and $\bigtriangledown\phi \neq 0$.)

Consider a Lagrangian for ϕ which *already* has the scale factor built into it as

$$\mathcal{L} = R^3[\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m^2\phi^2) - V(\phi)] \tag{4.49}$$

The R^3 factor comes from $\sqrt{-g} = R^3$ for a Robertson-Walker metric. This will be discussed in Chapter 7. *Notice that it is the same factor which sits outside the Friedmann Lagrangian in equation (2.20).* The equation of motion is **(do Problem 4.3)**

$$\ddot{\phi} - \bigtriangledown^2\phi + 3H\dot{\phi} + m^2\phi + V' = 0 \tag{4.50}$$

which is *different* to (4.45). (NNNN why ???) However if $m = 0$ and $\bigtriangledown\phi = 0$ it is the same as (4.46).

Let's only consider

$$\mathcal{L} = R^3[\frac{1}{2}\dot{\phi}^2 - V(\phi)] \tag{4.51}$$

which results from setting $m = 0$ and $\bigtriangledown\phi = 0$ in (4.49). The equation of motion is

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{4.52}$$

Notice how *quickly* we obtained this result rather than the long procedure to get (4.46). We didn't even use the energy-momentum tensor. Also realize that because $\bigtriangledown\phi = 0$ the above Lagrangian formalism is really no different to our old fashioned formalism where we had $q_i(t)$. Here we have only $\phi = \phi(t)$ (not $\phi = \phi(\mathbf{x})$), and so we only have $i = 1$, i.e. $q_i \equiv \phi$.

Identifying the Lagrangian as [29] $\mathcal{L} = R^3(T - V)$ we immediately write down the total energy density $\rho = T + V = \frac{1}{2}\dot{\phi}^2 + V(\phi)$. Taking the time derivative $\dot{\rho} = \dot{\phi}\ddot{\phi} + V'\dot{\phi} = -3H\dot{\phi}^2$ from (4.46) and substituting into the conservation equation (2.1), $\dot{\rho} = -3H(\rho + p)$ we obtain the pressure as $p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$. Thus our energy density and pressure derived here agree with our results above (4.39) and (4.40). Notice that the pressure is nothing more than $p = \frac{\mathcal{L}}{R^3}$. [29].

4.7.2 Limiting solutions

Assuming that $k = \Lambda = 0$ the Friedmann equation becomes

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}(\frac{1}{2}\dot{\phi} + V) \qquad (4.53)$$

This equation together with equation (4.46) form a set of coupled equations where solutions give $\phi(t)$ and $R(t)$. We solve the coupled equations in the standard way by first eliminating one variable, then solving one equation, then substituting the solution back into the other equation to solve for the other variable. Let's write equation (4.46) purely in terms of ϕ by eliminating R which appears in the form $H = \frac{\dot{R}}{R}$. We eliminate R by substituting H from (4.53) into (4.46) to give

$$\boxed{\ddot{\phi} + \sqrt{12\pi G(\dot{\phi}^2 + 2V)}\dot{\phi} + V' = 0}$$

$$\ddot{\phi}^2 + 2\ddot{\phi}V' - 12\pi G(\dot{\phi}^2 + 2V)\dot{\phi}^2 + V'^2 = 0 \qquad (4.54)$$

Notice that this is a *non-linear* differential equation for ϕ, which is difficult to solve in general. In this section we shall study the solutions for certain limiting cases. Once ϕ(t) is obtained from (4.54) it is put back into (4.53) to get $R(t)$.

Potential Energy=0

Setting $V = 0$ we then have $\rho = \frac{1}{2}\dot{\phi}^2 = p$. Thus our equation of state is

$$p = \rho \qquad (4.55)$$

or $\gamma = 3$.

With $V = V' = 0$ we have

$$\ddot{\phi}^2 + \sqrt{12\pi G}\dot{\phi}^2 = 0 \qquad (4.56)$$

which has the solution (**do problem 4.4**)

$$\phi(t) = \phi_o + \frac{1}{\sqrt{12\pi G}} \ln[1 + \sqrt{12\pi G}\dot{\phi}(t - t_o)] \qquad (4.57)$$

(Note that the solution is equation (9.18) of [29] is wrong.) Upon substituting this solution back into the Friedmann equation (4.53) and solving the differential equation we obtain (**do problem 4.5**)

$$R(t) = R_o[1 + \sqrt{12\pi G}\dot{\phi}_o(t - t_o)]^{1/3}. \tag{4.58}$$

This result may be understood from another point of view. Writing the Friedmann equations as

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho \tag{4.59}$$

and

$$\rho = \frac{\alpha}{R^m} \tag{4.60}$$

then the solution is always

$$R \propto t^{2/m} \tag{4.61}$$

which *always* gives

$$\rho \propto \frac{1}{t^2}. \tag{4.62}$$

If

$$\rho = \text{constant} \tag{4.63}$$

(corresponding to $m = 0$) then the solution is

$$R \propto e^t \tag{4.64}$$

(**do problem 4.6**). Note that for $m < 2$, one obtains power law inflation. For ordinary matter ($m = 3$), or radiation ($m = 4$) we have $R \propto t^{2/3}$ and $R \propto t^{1/2}$ respectively. Returning to the scalar field solution (4.57) the density is $\rho = \frac{1}{2}\dot{\phi}^2$ for $V = 0$. Thus

$$\dot{\phi}(t) = \frac{\dot{\phi}_o}{1 + \sqrt{12\pi G}\dot{\phi}_o(t - t_o)} \tag{4.65}$$

combined with $(\frac{R}{R_o})^3 = 1 + \sqrt{12\pi G}\dot{\phi}_o(t - t_o)$ from (4.58) yields

$$\dot{\phi}(t) = \frac{\dot{\phi}_o R_o^3}{R^3} \tag{4.66}$$

to give the density

$$\boxed{\rho = \frac{1}{2}\frac{\dot{\phi}_o^2 R_o^6}{R^6}} \tag{4.67}$$

corresponding to $m = 6$ and thus $R \propto t^{1/3}$ in agreement with (4.58). Note also that this density $\rho \propto \frac{1}{R^6}$ also gives $\rho \propto \frac{1}{t^2}$.

Thus for a scalar field with $V = 0$, we have $p = \rho$ ($\gamma = 3$) and $\rho \propto \frac{1}{R^6}$. Contrast this with matter for which $p = 0 (\gamma = 0)$ and $\rho \propto \frac{1}{R^3}$ or radiation for which $p = \frac{1}{3}\rho$ ($\gamma = 1/3$) and $\rho \propto \frac{1}{R^4}$.

However equation (4.67) may *not* be interpreted as a decaying Cosmological Constant because $p \neq \rho$ (see later).

Kinetic Energy=0

Here we take $\dot{\phi} = \ddot{\phi} = 0$, so that $\rho = V$ and $p = -V$ giving

$$p = -\rho \tag{4.68}$$

or $\gamma = -3$ which is a *negative pressure* equation of state. Our equation of motion for the scalar field (4.54) becomes

$$V' = 0 \tag{4.69}$$

meaning that

$$V = V_o \tag{4.70}$$

which is *constant*. Substituting the solution into the Friedmann equation (4.53) gives

$$H^2 = (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3} V_o \tag{4.71}$$

which acts as a *Cosmological Constant* and which has the solution (**do problem 4.7**)

$$R(t) = R_o e^{\sqrt{\frac{8\pi G}{3}V_o}(t-t_o)} \tag{4.72}$$

which is an inflationary solution, valid for *any* V.

Warning

We have found that if $k = \Lambda = 0$ and if $\rho \propto \frac{1}{R^m}$ then $R \propto t^2$ for *any* value of m. All of this is *correct*. To check this we might substitute into the Friedmann equation as

$$H^2 = (\frac{\dot{R}}{R})^2 \propto \frac{1}{t^2} \tag{4.73}$$

and say $\frac{\dot{R}}{R} \propto \frac{1}{t}$ giving $\int \frac{1}{R}\frac{dR}{dt}dt \propto \int \frac{dt}{t}$ which yields $\ln R \propto \ln t$ and thus $R \propto t^{2/m}$. The result $R \propto$t is wrong because we have left out an important constant.

Actually if $\frac{\dot{R}}{R} = \frac{c}{t}$ then $\ln R = c \ln t = \ln t^c$ giving $R \propto t^c$ instead of R$\propto$t.

Let's keep our constants then. Write $\rho = \frac{d^2}{R^m}$ then $R = (\frac{md}{2})^{2/m} t^{2/m}$ and $\rho = \frac{d^2}{(\frac{md}{2})^2 t^2} = \frac{(2/m)^2}{t^2}$. Substituting into the Friedmann equation gives $(\frac{\dot{R}}{R})^2 = \frac{(2/m)^2}{t^2}$ or $\frac{\dot{R}}{R} = \frac{(2/m)}{t}$ with the above constant $C = \frac{2}{m}$ yielding $R \propto$t$^{2/m}$ in *agreement* with the correct result above.

The lesson is be careful of constants when doing back-of-the-envelope calculations.

4.7.3 Exactly Solvable Model of Inflation

Because (4.54) is a difficult non-linear equation, exactly solvable models are very rare. We shall examine the model of Barrow [35] which can be solved exactly and leads to power law inflation. The advantage of an exactly solvable model is that one can develop ones physical intuition better. Barrow's model [35] is briefly introduced by Islam [13].

Any scalar field model is specified by writing down the potential $V(\phi)$. Barrow's potential is

$$V(\phi) \equiv \beta e^{-\lambda\phi} \tag{4.74}$$

where β and λ are constants to be determined. Barrow [35] claims that a *particular solution* to (4.54) is (which was presumably guessed at, rather then solving the differential equation)

$$\phi(t) = \sqrt{2A} \ln t \tag{4.75}$$

where $\sqrt{2A}$ is just some constant. We check this claim by substituting (4.74) and (4.75) into (4.54). From this we find (do problem 4.9) that

$$\lambda = \sqrt{\frac{2}{A}} \tag{4.76}$$

and

$$\beta = -A \tag{4.77}$$

or

$$\beta = A(24\pi G A - 1) \tag{4.78}$$

Note that Barrow is *wrong* when he writes $\lambda A = \sqrt{2}$. Also he uses units with $8\pi G = 1$, so that the second solution (4.78), he writes correctly as $\beta = A(3A-1)$. Also Barrow doesn't use the first solution (4.77) for reasons we shall see shortly.

Having solved for $\phi(t)$ we now substitute into (4.53) to solve for $R(t)$. (Recall $\phi(t)$ and $R(t)$ are the solutions we seek to our coupled equations (4.46) and (4.53).) Substituting $V = \frac{\Lambda}{t^2}$ and $\dot{\phi} = \frac{\sqrt{2A}}{t}$ (see solution to problem 4.9) we have

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}(\frac{1}{2}\frac{2A}{t^2} + \frac{\beta}{t^2}) = \frac{8\pi G}{3}(A+\beta)\frac{1}{t^2} \tag{4.79}$$

giving an equivalent density

$$\rho = \frac{A+\beta}{t^2} \tag{4.80}$$

Clearly we see why we reject the first solution (4.77) with $\beta = -A$. It would give zero density. Using the second solution (4.78) with $\beta = A(24\pi GA - 1)$ yields

$$\rho = \frac{24\pi GA^2}{t^2}. \tag{4.81}$$

Solving the Friedmann equation (4.79) gives

$$R \propto t^{8\pi GA} \tag{4.82}$$

where D is some constant. Setting $8\pi G \equiv 1$ we have

$$R \propto t^A \tag{4.83}$$

in agreement with Barrow's solution. Power law inflation results for

$$A > 1. \tag{4.84}$$

Inverting the solution (4.83) we have $t^2 = C' R^{2/A}$ where C' is some constant. Substituting into (4.81) we have

$$\rho \propto \frac{1}{R^{2/A}} \tag{4.85}$$

which corresponds to a *Weak decaying Cosmological Constant.* (See sections 4.7.4 and 4.7.5) For the inflationary result $A > 1$ we have $\frac{2}{A} \equiv m < 2$ which corresponds to the quantum tunneling solution!!

Note of course that (4.85) can also be obtained via $\rho = \frac{1}{3}\dot{\phi}^2$+V. We have V=$\frac{\beta}{t^2} \propto \frac{1}{R^{2/A}}$ and $\dot{\phi} = \frac{\sqrt{2A}}{t}$ giving $\dot{\phi}^2 \propto \frac{1}{R^{2/A}}$.

4.7.4 Variable Cosmological Constant

In this section we address the question as to when the density can be interpreted as a Cosmological Constant. Recall the Friedmann equations

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3} \tag{4.86}$$

and

$$\begin{aligned} \frac{\ddot{R}}{R} &= -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \\ &= -\frac{4\pi G}{3}\rho(1+\gamma) + \frac{\Lambda}{3} \end{aligned} \tag{4.87}$$

for $p = \frac{\gamma}{3}\rho$. Suppose $\rho = k = 0$, then we have

$$H^2 = (\frac{\dot{R}}{R})^2 = \frac{\Lambda}{3} \tag{4.88}$$

and

$$\frac{\ddot{R}}{R} = \frac{\Lambda}{3} \tag{4.89}$$

where two things have happened. Firstly the velocity and acceleration equations both have the same right hand side. Secondly the acceleration is positive. What sort of density would give the same result. Again for $k = 0$

$$H^2 = (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho \tag{4.90}$$

and

$$\frac{\ddot{R}}{R} = \frac{8\pi G}{3}\rho \tag{4.91}$$

only for $\gamma = -3$ or $p = -\rho$. From our conservation equation, $\dot{\rho} = -3H(\rho+p)$, this can only happen for ρ=constant. *Thus constant density with equation of state $p = -\rho$ acts identically to a Cosmological Constant. In addition the solution is automatically are of exponential inflation, $R \propto e^{Ht}$.* (Exercise:verify this.) Let us define a *Strong Cosmological Constant* as one in which *the velocity and acceleration equations both have the same right hand side*, (which automatically implies that the acceleration is positive). Such a Strong Cosmological Constant must be a true constant.

On the other hand we can imagine densities that still give a positive acceleration (i.e. inflation) but do *not* normally give the velocity and acceleration with the same right hand side. Examining (4.54) indicates that the *acceleration is guarenteed to be positive if* $\gamma < -1$ giving $p < -\frac{1}{3}\rho$. (Recall that the exponential inflation above required $\gamma = -3$, which is consistent with the inequality.) Thus negative pressure gives inflation. (Although not *all* negative pressure gives inflation, e.g. $p = -\frac{1}{4}\rho$.) The inflation due to $\boxed{\gamma < -1}$ will not be exponential inflation, but something weaker like perhaps power law inflation. Let us define a *Weak Cosmological Constant* as one which arises from negative pressure (actually $p < -\frac{1}{3}\rho$) to give a positive acceleration (inflation) only. The velocity and acceleration equations need *not* have the same right hand side.

Recall that ordinary matter and radiation, or any positive pressure equation of state, neccesarily leads to negative acceleration (with $\Lambda = 0$). Thus *positive pressure leads to attractive gravity.* However positive acceleration implies a repulsive gravity or antigravity. Thus *negative pressure (actually $p < -\frac{1}{3}\rho$) leads to antigravity.* This is why we wish to use the term weak Cosmological Constant (even though right hand sides are not the same) because it is consonant with antigravity.

Let us *summarize*. We consider ρ *alone* with $\Lambda \equiv 0$. For

$$\begin{aligned} \ddot{R} &> 0 \Rightarrow \gamma < -1 \ \text{ and } \rho = \text{anything (e.g. } \rho = \rho(\text{R}) \text{ or } \rho = \text{constant)} \\ \ddot{R} &> 0 \ \text{ and } \rho = \text{constant} \Rightarrow \gamma = -3 \end{aligned}$$

For $\gamma < -1$, ρ behaves as a weak Cosmological Constant and for $\gamma = -3$, ρ behaves as a Strong Cosmological Constant.

Finally let us emphasize that it is perfectly legitimate to consider a Weak Cosmological Constant as a *real* Cosmological Constant. Einstein's original motivation in introducing Λ was to obtain a static universe. Thus all he wanted was a antigravity term; i.e. all he wanted was a weak Cosmological Constant. It "accidentally" happened that the right hand sides turned out to be equal, giving a strong Λ.

Interpreting ρ as a Cosmological Constant leads us to expect that a weak Cosmological Constant can vary. This follows from ρ=anything above. i.e. ρ=constant or $\rho = \rho(R)$ giving a variable function. (But a strong Cosmological Constant cannot vary).

4.7.5 Cosmological constant and Scalar Fields

Refer back to the density and pressure of the scalar field in equations (4.41)) and (4.42). We had

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

and

$$\rho = \frac{1}{2}\dot{\phi}^2 - V(\phi).$$

For the case where $PE = V = 0$ we have $p = \rho$. The pressure is positive and therefore equation (4.67), $\rho \propto \frac{1}{R^6}$, *cannot* be interpreted as a (variable) Cosmological Constant.

For the case where $KE = \frac{1}{2}\dot{\phi}^2 = 0$ we have $p = -\rho$ meaning that ρ *can* be interpreted as a Strong Cosmological Constant. (We found $V' = 0$ and thus $\rho = V = constant = V_o$ and $p = -V_o$.)

These results are true in general (assuming $m = \nabla\phi = 0$) as we have not yet specified $V(\phi)$.

Let us now consider the Barrow model $V(\phi) = \beta e^{-\lambda\phi}$. we found that $\phi(t) = \frac{2}{\lambda}\ln t$ and $\beta = \frac{2}{\lambda^2}(\frac{6}{\lambda^2} - 1)$ for $8\pi G \equiv 1$. Introducing $A \equiv \frac{2}{\lambda}$ we can re-write as $\beta = A(3A - 1)$ and $\phi(t) = \sqrt{2A}\ln t$ and $V(\phi) = A(3A-1)e^{\phi\frac{2}{A}}$. Substituting we obtain

$$\rho_{Barrow} = \frac{3A^2}{t^2} \tag{4.92}$$

and

$$\rho_{Barrow} = \frac{3A^2}{t^2}(\frac{2A}{3} - 1) = (\frac{2A}{3} - 1)\rho \tag{4.93}$$

the general equation of state is $p = \frac{\gamma}{3}\rho$ giving the Barrow equation of state

$$\gamma_{Barrow} = \frac{2}{A} - 3 \tag{4.94}$$

in equation (4.84) we concluded that power law inflation results for $A > 1$. Substituting this into (4.94) implies

$$\gamma_{Barrow} < -3 \tag{4.95}$$

which we *expect* because power law inflation implies $\ddot{R} > 0$. thus for $A > 1$ the Barrow pressure is negative with $\gamma < -1$ and thus $\rho_{Barrow} = \frac{3A^2}{t^2}$ corresponds to a Weak Cosmological Constant. Furthermore this Cosmological Constant is variable and decays with time. in equation (4.85) we wrote this as $\rho_{Barrow} \propto \frac{1}{R^{2/A}}$.

4.7.6 Clarification

We wish to clarify the distinction between a *Strong* Cosmological Constant, a *Weak* Cosmological Constant and a *Varying* Cosmological Constant. A Strong Cosmological Constant occurs when the Friedmann equation are of the form

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{\Lambda}{3} \tag{4.96}$$

and

$$-qH^2 \equiv \frac{\ddot{R}}{R} = \frac{\Lambda}{3} \tag{4.97}$$

and comparing with

$$(\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho \tag{4.98}$$

$$\frac{\ddot{R}}{R} = -\frac{8\pi G}{3}(\rho + 3p) \tag{4.99}$$

leads us to conclude that the equation of state for a Strong Cosmological Constant is $p = -\rho$, where we have identified $\Lambda \equiv 8\pi \mathrm{G}\rho_v$. The conservation equation

$$\dot{\rho} = -3H(\rho + p) \tag{4.100}$$

leads us to conclude that $\Lambda \equiv 8\pi G\rho_v$=constant for a Strong Cosmological Constant.

A *Variable* Cosmological Constant $\Lambda(R)$, can also be considered but the equation *cannot* be like (4.96) and (4.97) with no ρ term. As we shall see below a Varying Cosmological Constant automatically involves matter creation (existence of a ρ term). Thus Λ(R) cannot be written down in the Friedmann equation without *also* writing ρ. Thus for a Varying Cosmological Constant (let $k = 0$) we have

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda(R)}{3} \tag{4.101}$$

and

$$-qH^2 \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(1+\gamma)\rho + \frac{\Lambda}{3} \tag{4.102}$$

One can easily show that the conservation equation becomes

$$\boxed{\dot{\rho} + \dot{\rho}_v = \text{-3H}(\rho\text{+p})} \quad = -\mathrm{H}(3+\gamma)\rho$$

(4.103)

(**do Problem 4.10**). Other ways of writing this are (see equations 2.1-2.7).

$$\frac{d(\rho R^3)}{dt} + p\frac{dR^3}{dt} + R^3\dot{\rho}_v = 0 \tag{4.104}$$

$$\frac{d(\rho R^{3+\gamma})}{dt} + R^{3+\gamma}\dot{\rho}_v = 0 \tag{4.105}$$

$$\rho' + \rho_v' = -\frac{3}{R}(\rho + p) \tag{4.106}$$

$$\frac{d(\rho R^3)}{dR} + 3pR^2 + R^3\rho_v' = 0 \tag{4.107}$$

$$\frac{1}{R^{3+\gamma}}\frac{d(\rho R^{3+\gamma})}{dR} = -\rho_v' \tag{4.108}$$

The last equation show clearly that if $\rho_v \neq 0$ (i.e. $\rho_v \neq$ constant) then matter or radiation is created or destroyed. This is why we cannot write (4.96) and (4.97) if Λ is allowed to vary. We must include particle creation terms as in (**??**) and (4.102).

For a Variable Cosmological Constant the same term $\frac{\Lambda(R)}{3}$ appears in both right hand sides of the Friedmann equation (4.101) and (4.102). However we don't know its equation of state. Also γ can be anything and thus $\dot{\rho}_v$ can lead to creation of matter, radiation or anything else. Further $\Lambda(R)$ can vary.

Both a Strong and Weak Cosmological Constant have well defined equation of state, both with (different) negative pressure. A Weak Cosmological Constant can vary but the right hand sides of the Friedmann equations differ.

Thus we see that we have 3 different types of Cosmological Constant namely, *Strong*, *Weak*, and *Variable*, each with both similar and dissimiliar properties. These are summarize in the Table.

4.7.7 Generic Inflation and Slow-Roll Approximation

We shall first discuss features that are common to many models of inflation based on scalar fields [21]. These models typically have a large region of the potential where the potential is flat (slow-roll region) and then a harmonic

	Weak	Strong	Variable
Right Hand Sides of Velocity and Acceleration Eqns.	Different	same	Same (but must include ρ)
Eqn. State	$p < -\frac{1}{3}\rho$	$p = -\rho$	Unknown
$\Lambda = 8\pi G\rho_v$	Variable	Constant	Variable

Table 4.1: Properties of 3 types of Cosmological Constants

region where the potential looks similar to an oscillatior (rapid oscillation region). Such a potential is shown in Fig 4.1.

Let us first analyze this situation from a simple physics point of view. If Fig 4.1 were actually a plot of $V(x)$ versus x the equation of motion would be

$$m\ddot{x} + b\dot{x} + V' = 0 \tag{4.109}$$

which, for $V = \frac{1}{2}kx^2$ or $V' = kx$ would represent a damped harmonic oscillator, which indeed is the case in the rapid oscillation region. In the slow roll regime we have $V \approx$constant or $V = V_o - Cx$ where C is small, giving $V' = -C$. The physical situation would represent a *ball rolling down a hill with friction into a valley* [21].

In the slow roll regime friction dominates and the ball moves at terminal velocity [21], $\dot{x}$=constant and $\dot{x}$ doesn't change much so that $\ddot{x} = 0$. This is the slow roll regime we have

$$b\dot{x} + V' = 0 \tag{4.110}$$

and for $V = V_o - Cx$ we have

$$b\dot{x} \approx C \tag{4.111}$$

When the ball finally reaches the valley we are back to the full equation (4.109) where $V' \equiv kx$ so that the ball experiences rapid, damped oscillations.

Let's return to the scalar field equations which are (for $\Lambda = k = 0$)

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}(\frac{1}{2}\dot{\phi} + V) \tag{4.112}$$

and

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{4.113}$$

where the friction term $3H$ is due to the expansion of the universe [21]. Based on analogy with the rolling ball, the slow roll approximation ($\dot{\phi} \approx$constant, $\ddot{\phi} \approx 0$) is

$$3H\dot{\phi} + V' \approx 0 \tag{4.114}$$

In *addition*, the slow roll region is characterized by a small kinetic energy $\frac{1}{2}\dot{\phi}^2 << V$, so that

$$H^2 \approx \frac{8\pi G}{3} V_o \approx constant \tag{4.115}$$

For the rolling ball we only had one equation, but here we have two equations (4.114) and (4.115) involved in the slow roll approximations. The slow roll equation can also be obtained directly from equation (4.54) where we set $\dot{\phi}^2 << 2V_o$ and $\ddot{\phi} \approx 0$ to give

$$\sqrt{24\pi G V_o}\dot{\phi} + V' = 0 \tag{4.116}$$

which is consistent with combining (4.114) and (4.115). Normally we would have the formula for V(ϕ) and solve (4.116) or (4.54) first for $\phi(t)$ and then substitute our answer into the Friedmann equation to obtain R(t). However we have not yet specified $V(\phi)$. We *have* specified that we are in a slow roll regime thus making (4.115) valid which we can solve directly for $R(t)$ giving an exponential inflationary solution

$$R(t) {\approx} R_o e^{t\sqrt{\frac{8\pi G}{3}V_O}} \tag{4.117}$$

Thus, no matter what $V(\phi)$ is, the slow regime always give approximately exponential inflation.

To obtain $\phi(t)$ we must specify $V(\phi)$ even in the slow-roll regime.

4.7.8 Chaotic Inflation in Slow-Roll Approximation

The Chaotic inflation model is defined by the potential

$$V(\phi) = \frac{1}{2}m^2\phi^2 \tag{4.118}$$

which is exactly analogous to the harmonic oscillator potential $V(x) = \frac{1}{2}kx^2$.

Using $V_o = \frac{1}{2}m^2\phi_o^2$ the slow roll solution (4.117) becomes

$$\begin{aligned} R(t) &= R_o e^{t\sqrt{\frac{4\pi}{3}}\sqrt{G}m\phi} \\ &= R_o e^{t\sqrt{\frac{4\pi}{3}}\frac{m}{M_p}\phi} \end{aligned} \tag{4.119}$$

where $G \equiv \frac{1}{M_p^2}$ is in agreement with equation (9.23) of Madsen [29]. Having specified $V(\phi)$ in (4.118) we can solve (4.116), which

$$\dot{\phi} = -\frac{1}{\sqrt{24\pi G}}\frac{V'}{\sqrt{V_o}} \tag{4.120}$$

or

$$\frac{\dot{\phi}}{\phi} = -\frac{m}{\sqrt{12\pi G}}\frac{1}{\phi_o} \tag{4.121}$$

This gives

$$\begin{aligned} \phi(t) &= \phi_o e^{-\frac{m}{\phi_o\sqrt{12\pi G}}t} = \phi_o e^{-\frac{mM_p}{\phi_o\sqrt{12\pi}}t} \\ &= \phi_o - \frac{mM_p}{\sqrt{12\pi}}t \end{aligned} \tag{4.122}$$

is the slow roll region in agreement with equation (9.25) of Madsen [29]. For $m << M_p$ or for short times we see that $\phi(t)$ will be approximately constant, or slowing rolling. We see that $\phi(t)$ is a decaying exponential in time. Thus there will be a "half-life" or "lifetime" associated with slow roll which we define as

$$\tau = \frac{\phi_o\sqrt{12\pi}}{mM_p} \tag{4.123}$$

When $t = \tau$ we see that $\phi = \phi_o\frac{1}{e}$. That is the amplitude is reduced by the factor $\frac{1}{e}$ We expect that the slow roll approximation will be valid for $t < \tau$.

Our solution of ϕ(t) in (4.122) came from solving (4.116). We can also obtain $\phi(t)$ slightly differently. Let's *not* assume $V = V_o$=constant, but only that $\dot{\phi}^2 << 2V$ and $\ddot{\phi} \approx 0$. Then instead of (4.116) we have

$$\sqrt{24\pi GV}\dot{\phi} + V' = 0 \tag{4.124}$$

which, for chaotic inflation becomes

$$\sqrt{12\pi G}m\phi\dot{\phi} + m^2\phi = 0 \tag{4.125}$$

or

$$\dot{\phi} = -\frac{m}{\sqrt{12\pi G}} \tag{4.126}$$

which has the solution

$$\begin{aligned} \phi(t) &= \phi_o - \frac{m}{\sqrt{12\pi G}}t \\ &= \phi_o - \frac{mM_p}{\sqrt{12\pi}}t \end{aligned} \tag{4.127}$$

in agreement with (4.122) for short time.

We can further investigate the validity of the slow roll approximation by evaluating the potential as a function of time and checking that it is constant for short times. We do this by substituting our solution for $\phi(t)$ back into the potential. We get

$$\begin{aligned} V &= \frac{1}{2}m^2\phi^2 = \frac{1}{2}m^2\phi_o^2 e^{-\frac{mM_p}{\phi_o\sqrt{3\pi}}t} \\ &\approx \frac{1}{2}m^2\phi_o(\phi_o - \frac{mM_p}{\sqrt{3\pi}}t) \end{aligned} \tag{4.128}$$

Thus we see that for $m << M_p$ or for short times the potential is indeed constant.

For short times (or for $m << M_p$) we have verified that ϕ and V are approximately constant. This means that $\dot{\phi} \approx 0$ and $\rho \approx V_o$ which give exponential inflation. (Also $p \approx -V_o$, so that $p = -\rho$).

In order to solve the horizon, flatness and monopole problems, most models require a high degree of inflation typically amounting to about 60 e-folds [21]. Given $R(t) = R_o e^{Ht}$, the number of e-folds after time t is

$$N = \ln(\frac{R(t)}{R_o}) = Ht. \tag{4.129}$$

(Actually a better formula is given in equation (8.26) of the book by Kolb and Turner [21] (pg.278)). After one lifetime τ, the number of e-folds is

$$N = H\tau = \sqrt{\frac{4\pi}{3}}\frac{m}{M_p}\phi_o\frac{\phi_o\sqrt{12\pi}}{mM_p} = 4\pi\frac{\phi_o^2}{M_p^2} \tag{4.130}$$

where $H = \sqrt{\frac{4\pi}{3}}\frac{m}{M_p}\phi_o$ taken from (4.119) and τ is from (4.123). Thus the requirement $N \geq 60$ yields

$$N \geq 60 \quad \Rightarrow \quad \phi_o \geq \sqrt{5}M_p. \tag{4.131}$$

Notice how the flatness problem is solved in inflation. We have $H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} \approx \frac{8\pi G}{3}V_o - \frac{k}{R^2}$. During inflation (slow roll) V_o stays constant but by the end of inflation $R = R_o e^{60}$ or $\frac{1}{R^2} = \frac{1}{R_o^2}e^{-120}$. The term $\frac{k}{R^2}$ has dropped by e^{-120}, whereas V_o has remained constant. Thus the term $\frac{k}{R^2}$ is entirely negligible. Inflation does *not* give $k = 0$, but rather gives $\frac{k}{R^2} \approx 0$ which is equivalent to $k = 0$. This is an improtant distinction. The universe can have $k = 0$ or $k = +1$ or $k = -1$. No matter what the value of k, it gets diluted by inflation and is equivalent to a universe with $k = 0$. thus *within our horizon the universe is flat.* Quantum cosmology predict that a universe which arises via tunnelling must have $k = +1$. This is perfectly OK with inflation which simply dilutes the curvature.

Quantum tunnelling requires $k = +1$. Inflation actually says *nothing* about the value of k. It simply predicts that $\frac{k}{R^2} \approx 0$ at the end of inflation.

On Earth, the reason many people believe the Earth is flat is because we cannot see beyond the horizon. Up to the horizon it *looks* flat. If we *could* see beyond the horizon we would see the curvature. Similarly for our universe. According to inflation the size of the universe is *much* larger then the distance to the horizon (i.e. as far as we can see) the universe looks flat becuase $\frac{k}{R^2}$ is negligible. If we could see *beyond* the horizon we would see the curvature. And quantum tunnelling predicts that what we would see would be a universe of positive curvature.

Cosmological Constant associated with Chaotic Inflation

Let us now calculate the density as a function of R. We solve (4.119) for $t = t(R)$ as

$$t = \sqrt{\frac{3}{4\pi}}\frac{M_p}{m\phi_o}\ln(\frac{R}{R_o}). \tag{4.132}$$

Substituting into (4.122) we have

$$\phi(R) = \phi_o(\frac{R_o}{R})^{\frac{M_p^2}{4\pi\phi_o^2}} \tag{4.133}$$

and with $V(\phi) = \frac{1}{2}m^2\phi^2$ we have

$$V(R) = \frac{1}{2}m^2\phi_o^2(\frac{R_o}{R})^{\frac{M_p^2}{2\pi\phi_o^2}}. \tag{4.134}$$

Also from (4.122)

$$\dot{\phi}(t) = -\frac{mM_p}{\sqrt{12\pi}}e^{-\frac{mM_p}{\phi_o\sqrt{12\pi}}t} \tag{4.135}$$

so that

$$\dot{\phi}(R) = -\frac{mM_p}{\sqrt{12\pi}}(\frac{R_o}{R})^{\frac{M_p^2}{4\pi\phi_o^2}}. \tag{4.136}$$

Finally evaluating $\rho = \frac{1}{2}\dot{\phi}^2 + V$ we have

$$\rho(R) = \frac{1}{2}m^2(\frac{M_p^2}{12\pi} + \phi_o^2)(\frac{R_o}{R})^{\frac{M_p^2}{2\pi\phi_o^2}}. \tag{4.137}$$

Thus

$$\rho(R) \propto \frac{1}{R^m} \quad \text{where} \quad m \equiv \frac{M_p^2}{2\pi\phi_o^2}. \tag{4.138}$$

Recall previously that an inflationary solution requires $m < 2$, yielding

$$\phi_o > \frac{M_p}{\sqrt{4\pi}} = 0.3\ M_p \tag{4.139}$$

which is entirely consistent with (4.131)!

What is the equation of state? using $p = \frac{1}{2}\dot{\phi}^2 - V$ we obtain

$$p = \frac{1}{2}m^2(\frac{M_p^2}{12\pi} - \phi_o^2)(\frac{R_o}{R})^{\frac{M_p^2}{2\pi\phi_o^2}}. \tag{4.140}$$

The question is, is the pressure negative? We find that for inflation to occur we need $\phi_o > \frac{M_p}{\sqrt{4\pi}}$. Write this as

$$\phi_o = \ell\frac{M_p}{\sqrt{4\pi}} \quad \textit{with} \quad \ell > 1 \tag{4.141}$$

Thus the density becomes

$$\rho(R) = \frac{m^2M_p^2}{8\pi}(\frac{1}{3} + \ell^2)(\frac{R_o}{R})^m \tag{4.142}$$

and

$$p(R) = \frac{m^2M_p^2}{8\pi}(\frac{1}{3} - \ell^2)(\frac{R_o}{R})^m \tag{4.143}$$

Defining $\kappa \equiv \frac{m^2M_p^2}{8\pi}(\frac{R_o}{R})^m$, we write $\rho = (\frac{1}{3} + \ell^2)\kappa$ and $p = (\frac{1}{3} - \ell^2)\kappa$, giving $p = \frac{1-3\ell^2}{1+3\ell^2}\rho$. The requirement $\ell > 1$ yields

$$p < -\frac{1}{2}\rho \tag{4.144}$$

which means negative pressure! Writing $p = \frac{\gamma}{3}\rho$ gives

$$\gamma = -\frac{3}{2}. \tag{4.145}$$

These results are in agreement with our previous constraints that in order to have positive $\ddot{R}$ (inflation) we needed $p < \frac{1}{3}\rho$ or $\gamma < -1$.

Our chaotic inflation model is the slow roll approximation gives negative pressure (but not $p = -\rho$) and corresponds to a Weak Decaying Cosmological Constant!

4.7.9 Density Fluctuations

An important result that we shall use without proof is that fluctuations of the scalar field are given approximately by

$$\boxed{\delta\phi \approx \frac{H}{2\pi}} \tag{4.146}$$

this result is discussed by Linde [36] (Pg.17,50), Kolb and Turner [21](Pg.284), Collins, Martin and Squires [37] (Pg.410) and by Dolgov, Sazhin and Zeldovich [38].

Using $\rho = \frac{1}{2}\dot{\phi}^2 + \mathrm{V}(\phi)$ we have $\frac{d\rho}{d\phi} = V'(\phi)$ or

$$\delta\rho = V'\delta\phi \approx V'\frac{H}{2\pi} \tag{4.147}$$

which, upon assuming $\dot{\phi} = 0$, gives

$$\frac{\delta\rho}{\rho} = \sqrt{\frac{2}{3\pi}\frac{GV'^2}{V}} = \sqrt{\frac{2}{3\pi}}\frac{V'}{M_p\sqrt{V}} \tag{4.148}$$

where we have used $G \equiv \frac{1}{M_p^2}$. For the chaotic inflation model, $V = \frac{1}{2}m^2\phi^2$ this yields

$$\frac{\delta\rho}{\rho} = \sqrt{\frac{4}{3\pi}}\frac{m}{M_p}. \tag{4.149}$$

This is an intensely important formula often written as

$$\frac{\delta\rho}{\rho} \approx m\sqrt{G} = \frac{m}{M_p} \tag{4.150}$$

the density fluctuations observed by COBE are $\frac{\delta\rho}{\rho} \approx 10^{-5}$ yielding $m = 10^{-5}M_p$.

The above formula is not very useful for the Barrow model where a well defined inflation means $m \equiv m_{inflation}$ is not present. In that case the formula is written more usefully as [?, ?, ?]

$$\frac{\delta\rho}{\rho} = \frac{\rho_{inflation}}{\rho_p} \tag{4.151}$$

4.7.10 Equation of State for Variable Cosmological Constant

In this section we wish to demonstrate that Variable Cosmological Constant models have negative pressure [?, ?, ?].

Firstly if one *assumes*

$$p \equiv \frac{\gamma}{3}\rho \tag{4.152}$$

then the conservation law follows as

$$\frac{1}{R^{3+\gamma}}\frac{d}{dR}(\rho R^{3+\gamma}) = -\rho_v'. \tag{4.153}$$

Let's assume that

$$\rho_v \equiv \frac{\alpha}{R^m} = \rho_{vo}(\frac{R_o}{R})^m. \tag{4.154}$$

Integrating the conservation law we have

$$\rho = \frac{A}{R^{3+\gamma}} + \chi\rho_v \tag{4.155}$$

where

$$\chi \equiv \frac{m}{3+\gamma-m} \tag{4.156}$$

and A is a constant give by

$$A = (\rho_o - \chi\rho_{vo})R_o^{3+\gamma}. \tag{4.157}$$

The pressure is

$$p = \frac{\gamma}{3}\frac{A}{R^{3+\gamma}} + \frac{\gamma}{3}\frac{m}{3+\gamma-m}\rho_v \tag{4.158}$$

This looks like bad news. Assuming that ρ_v dominate over the first term at some stage of evolution, it looks like the pressure only get negative for m$> 3+\gamma$. However, there are two things to keep in mind. Firstly, the pressure p is not the pressure of radiation or matter or vacuum because $\rho = \frac{A}{R^{3+\gamma}} + \chi\rho$, and $p = \frac{\gamma}{3}\rho$. The pressure that we would want to be negative would be the vacuum pressure p_v, which we shall work out below. Secondly, the key point is not as much having p negative but rather having $\ddot{R}$ positive. The equation

$$-qH^2 \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{8\pi G}{3}\rho_v \tag{4.159}$$

can still give postive $\ddot{R}$ even if p is not negative, because the ρ_v term has to be considered. The Friedmann equation is

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}(\rho + \rho_v) - \frac{k}{R^2} \tag{4.160}$$

Let us evaluate the right hand sides of these two equations (4.159) and (4.160).

From (4.155) we have

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}[\frac{A}{R^{3+\gamma}} + (\chi + 1)\rho_v] - \frac{k}{R^2} \tag{4.161}$$

where

$$\chi + 1 = \frac{3+\gamma}{3+\gamma-m}. \tag{4.162}$$

Also *assuming* p= $\frac{\gamma}{3}\rho$ we have

$$-qH^2 \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(1+\gamma)(\frac{A}{R^{3+\gamma}} + \chi\rho_v) + \frac{8\pi G}{3}\rho_v \tag{4.163}$$

$$= -\frac{4\pi G}{3}[1+\gamma)\frac{A}{R^{3+\gamma}} + \frac{(3+\gamma)(m-2)}{3+\gamma-m}\rho_v]. \tag{4.164}$$

Define

$$\theta \equiv \frac{(3+\gamma)(m-2)}{3+\gamma-m} \tag{4.165}$$

we see that θ *is always negative for* $m < 2$! Thus if the vacuum term dominates equation (4.164) then $\ddot{R}$ will be positive for $m < 2$. This agrees with our previous consideration that if $\rho = \frac{\alpha}{R^m}$ dominates the velocity equation then $m < 2$ leads to inflation.

Note however that θ can be negative for other values of m as shown in Table 4.2.

m	$\theta(\gamma = 0)$	$\theta(\gamma = 1)$
0.5	-1.8	-1.7
1	-1.5	-1.3
1.5	-1.0	-0.8
2	0	0
2.5	3.0	1.33
3	∞	4
4	-6	∞
5	-4.5	-12
6	-4	-8

Table 4.2: θ as a function of γ and m

Having established that a decaying Cosmological Constant can lead to negative pressure, let us now work out the vacuum equation of state for a decaying Cosmological Constant. Looking at (4.161) let us define

$$\tilde{\rho} \equiv \frac{A}{R^{3+\gamma}} \tag{4.166}$$

and

$$\tilde{\rho}_v \equiv (\chi + 1)\rho_v = \frac{3+\gamma}{3+\gamma-m}\rho_v \tag{4.167}$$

so that

$$\rho + \rho_v = \tilde{\rho} + \tilde{\rho}_v \tag{4.168}$$

giving

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}(\rho + \rho_v) - \frac{k}{R^2} = \frac{8\pi G}{3}(\tilde{\rho} + \tilde{\rho}_v) - \frac{k}{R^2} \tag{4.169}$$

and from (4.164) we have

$$-qH^2 \equiv \frac{\ddot{R}}{R} \quad = \quad -\frac{4\pi G}{3}[(1+\gamma)\rho - 2\rho_v] \tag{4.170}$$

$$= \quad -\frac{4\pi G}{3}[(1+\gamma)\tilde{\rho} + (m-2)\tilde{\rho}_v] \tag{4.171}$$

which we would *like* to write as

$$-qH^2 \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\tilde{\rho} + 3\tilde{p}) - \frac{4\pi G}{3}(\tilde{\rho}_v + 3\tilde{p}_v). \tag{4.172}$$

This is achieved if we make the following defintions

$$\boxed{\tilde{p} \equiv \frac{\gamma}{3}\tilde{\rho}} \tag{4.173}$$

and

$$\boxed{\tilde{p}_v \equiv \frac{m-3}{3}\tilde{\rho}_v \equiv \frac{\gamma_v}{3}\tilde{\rho}_v}, \tag{4.174}$$

which is our *vacuum equation of state* for a decaying Cosmological Constant. We see that for m< 3 we have $\tilde{p}_v$ and γ_v negative. For a $m < 2$ we have $\gamma_v < -1$ which we saw previously is the condition for inflation assuming vacuum domination of the density and pressure.

It is also satisfying to note that the equation of state for the non-vacuum component (equation 4.173) is identical to the equation of state for a perfect fluid that we encountered for models without a Cosmological Constant.

Alternative Derivation

Our definition of $\tilde{\rho}$ and $\tilde{\rho}_v$ above are not unique. We present alternative definitions below which will give the same vacuum equation of state but different for the non-vacuum component.

In equation (4.161) the density is

$$\begin{aligned}
\rho + \rho_v &= \frac{A}{R^{3+\gamma}} + (\chi+1)\rho_v \\
&= (\rho_o - \chi\rho_{vo})(\frac{R_o}{R})^{3+\gamma} + (\chi+1)\rho_{vo}(\frac{R_o}{R})^m \\
&= \rho_o(\frac{R_o}{R})^{3+\gamma} + \rho_{vo}[(\chi+1)(\frac{R_o}{R})^m - \chi(\frac{R_o}{R})^{3+\gamma}]
\end{aligned} \tag{4.175}$$

We previoulsy defined $\tilde{\rho} \equiv \frac{A}{R^{3+\gamma}}$ and $\tilde{\rho}_v \equiv (\chi+1)\rho_v$. However we might alternatively define

$$\bar{\rho} \equiv \rho_o(\frac{R_o}{R})^{3+\gamma} \tag{4.176}$$

and

$$\begin{aligned}
\bar{\rho}_v &\equiv \rho_{vo}[(\chi+1)(\frac{R_o}{R})^m - \chi(\frac{R_o}{R}^{3+\gamma}] \\
&= (\chi+1)\rho_v - \chi\frac{\rho_{vo}}{\rho_o}\bar{\rho}
\end{aligned} \tag{4.177}$$

still obtaining

$$\rho + \rho_v = \tilde{\rho} + \tilde{\rho}_v = \bar{\rho} + \bar{\rho}_v. \tag{4.178}$$

Assuming $p = \frac{\gamma}{3}\rho$ we have (compare to equation 4.171)

$$\begin{aligned} -qH^2 \equiv \frac{\ddot{R}}{R} &= -\frac{4\pi G}{3}[(1+\gamma)\rho - 2\rho_v] \\ &= -\frac{4\pi G}{3}[(1+\gamma - m\frac{\rho_{vo}}{\rho_o})\bar{\rho} + (m-2)\bar{\rho}_v] \end{aligned} \tag{4.179}$$

(where we have used the second expression in 4.177), which we would *like* to write as

$$-qH^2 \equiv \frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\bar{\rho} + 3\bar{p}) - \frac{4\pi G}{3}(\bar{\rho}_v + 3\bar{p}_v). \tag{4.180}$$

This is achieved if we make the following definitions

$$\boxed{\bar{p} \equiv \frac{\gamma - m\frac{\rho_{vo}}{\rho_o}}{3}\bar{\rho} \equiv \frac{\Gamma}{3}\bar{\rho}} \tag{4.181}$$

and

$$\boxed{\bar{p}_v \equiv \frac{m-3}{3}\bar{\rho}_v = \frac{\gamma_v}{3}\bar{\rho}_v} \tag{4.182}$$

4.7.11 Quantization

All of our proceeding work with the scalar field was at the *classical* level. In this section we wish to consider quantum effects.

In section (2.4.2) we dervied the wheeler-DeWitt equation in minisuperspace approximation. We began with the Lagrangian in equation (2.20)

$$L = -\kappa R^3[(\frac{\dot{R}}{R})^2 - \frac{k}{R^2} + \frac{8\pi G}{3}(\rho + \rho_v)] \tag{4.183}$$

and identified the conjugate momentum $p \equiv \frac{\partial L}{\partial \dot{R}} = -2\kappa\dot{R}R$ and derived the Wheeler-DeWitt equation, after quantizing with $p \to i\frac{\partial}{\partial R}$, as

$$\{-\frac{\partial^2}{\partial R^2} + 4\kappa^2[kR^2 - \frac{8\pi G}{3}(\rho + \rho_v)R^4]\}\psi = 0. \tag{4.184}$$

Notice that our quantization 'didn't do anything to the density."

In the work that we have done in the present chapter we have made an effort to write the scalar field as a function of R, i.e. $\phi = \phi(R)$ and using $\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$ we have written as an effective density $\rho(R)$ for the scalar field. Our intention has been to simply insert this ρ(R) into the Wheeler-DeWitt equation (4.184). In our work on inflation we found that for $\rho \propto \frac{1}{R^m}$ dominating the Friedmann equation then inflation occurs for $m < 2$. If this density also dominates $\rho + \rho_v$ in the Wheeler-DeWitt equation, then a *tunnelling* potential will *only* be present for $m < 2$. Thus inflation and quantum tunnelling require the same condition. This leads us to the hypothesis that inflation and quantum tunnelling are identical! Or in other words, inflation is simply a *classical* description of quantum tunnelling. We call this hypothesis *Quantum Inflation*.

Quantum inflation is easy to validate for ordinary densities, either ρ or ρ_v, that behave like $\rho \propto \frac{1}{R^m}$. With our discussion of the scalar field we have written $\rho_\phi \propto \frac{1}{R^m}$ so it would seem that the idea of quantum inflation also works for scalar fields.

In our quantization procedure we "didn't do anything to the density." In terms of scalar fields then this quantization procedure is different to what other people do with quantization. The usual procedure [**?**, **?**, **?**, 20, 21] is to quantize ϕ and R seperately and arrive at a Wheeler- DeWitt equation in terms of both of these variables. Let us now study this procedure.

We begin with the Lagrangian

$$L = -\kappa R^3[(\frac{\dot{R}}{R})^2 - \frac{k}{R^2}] + 2\pi^2 R^3[\frac{1}{2}\dot{\phi}^2 - V(\phi)] \tag{4.185}$$

where $\kappa \equiv \frac{3\pi}{4G}$.

From this one can deduce that

$$(\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}(\frac{1}{2}\dot{\phi}^2 + V) - \frac{k}{R^2} \tag{4.186}$$

and also

$$\ddot{\phi} + 3H\dot{\phi} + V' = 0 \tag{4.187}$$

provided one uses $\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p)$ with $\rho = \frac{1}{2}\dot{\phi}^2 + V$ and $p = \frac{1}{2}\dot{\phi}^2 - V$. (NNN see FE)

The canonical momenta are

$$\Pi_R \equiv \frac{\partial L}{\partial \dot{R}} = -2\kappa R\dot{R} \tag{4.188}$$

and

$$\Pi_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = 2\pi^2 R^3 \dot{\phi}. \tag{4.189}$$

The Hamiltonian ($H = p_i \dot{q}_i - L$) becomes (using $\kappa = \frac{3\pi}{4G}$)

$$\begin{aligned} H &= \Pi_R \dot{R} + \Pi_\phi \dot{\phi} - L & (4.190) \\ &= -\kappa R^3 [(\frac{\dot{R}}{R})^2 + \frac{k}{R^2} - \frac{8\pi G}{3}(\frac{1}{2}\dot{\phi}^2 + V)] \equiv 0 & (4.191) \end{aligned}$$

where the result $H = 0$ is obtained by comparing the expression for the Hamiltonian to the Friedmann equation (4.186). This Hamiltonian is *exactly* analogous to the Hamiltonian we had in equation (2.22) where we had ρ instead of $\frac{1}{2}\dot{\phi}^2 + V$.

Writing H in terms of the conjugate momenta we have

$$H = -\kappa R^3 [\frac{\Pi_R^2}{4\kappa^2 R^4} + \frac{d}{R^2} - \frac{8\pi G}{3}(\frac{\Pi_\phi^2}{8\pi^4 R^6} + V)] = 0 \tag{4.192}$$

which, of course is also equal to zero. This Hamiltonian is inherited to equation (11.34), Pg.462 of the book by Kolb and Turner [21]. This equation is re-arranged as

$$\Pi_R^2 - \frac{3}{4\pi G}\frac{2}{R^2}\Pi_\phi^2 + \frac{9\pi^2}{4G^2}(kR^2 - \frac{8\pi G}{3}VR^4) = 0. \tag{4.193}$$

In order to compare to our signal Wheeler-DeWitt equation let's replace Π_ϕ with $\Pi_\phi = 2\pi^2 R^3 \dot{\phi}$, which results in

$$\Pi_R^2 + \frac{9\pi^2}{4G^2}[kR^2 - \frac{8\pi G}{3}(\frac{1}{2}\dot{\phi}^2 + V)R^4] = 0 \tag{4.194}$$

which is *exactly* analogous to our original Wheeler-DeWitt equation (2.24) where we had ρ instead of $\frac{1}{2}\dot{\phi}^2 + V$.

Equation (4.193) is quantized by making the replacements

$$\Pi_R \to -i\frac{\partial}{\partial R} \tag{4.195}$$

and

$$\Pi_\phi \to -i\frac{\partial}{\partial \phi} \tag{4.196}$$

and setting $H\psi = 0$ to give

$$\boxed{[-\frac{\partial^2}{\partial R^2} + \frac{3}{4\pi G}\frac{1}{R^2}\frac{\partial^2}{\partial \phi^2} + \frac{9\pi^2}{4G^2}(\mathrm{kR}^2 - \frac{8\pi G}{3}\mathrm{VR}^4)]\psi = 0} \tag{4.197}$$

which is the Wheeler-DeWitt equation in minisuperspace approximation for a quantized scalar field ϕ. This is *identical* to equation (10.1.11), Pg.270 of the book by Linde [36].

We identify the potential as

$$\boxed{U(R,\phi) = \frac{9\pi^2}{4G^2}(kR^2 - \frac{8\pi G}{3}VR^4)} \tag{4.198}$$

which is identical to equation (11.35), Pg.463 of the book by Kolb and Turner [21]. In equation (10.1.25), Pg277 of the book by Linde [36], he gives an expression for $V(R,\phi) = \frac{3\pi}{G}V(a)$, so that our result does agree with Linde.

We can see that the above method of quantizing the scalar field ϕ directly is still consistent with our idea of Quantum Inflation. Recall that $\rho(R)$ and $V(R)$ in terms of $\phi(R)$ obvioulsy ρ, V and $\dot{\phi}^2$ *must have the same* R *dependence*. Thus if $\rho \propto \frac{1}{R^m}$ then also $V \propto \frac{1}{R^m}$ in the same way. Thus our potential $U(R,\phi)$ will alsways exhibit a tunnelling shape for $m < 2$. Thus Quantum Inflation still works for $U(R,\phi)$ when ϕ is quantized seperately.

4.8 Problems

4.1 Show that $\partial_\mu j^\mu = 0$ is the equation of continuity $\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0$ where $j^\mu \equiv (\rho, \mathbf{j})$.

4.2 Show that the above equation of continuity *also* results from taking the divergence of Ampèrés law.

4.3 Show that the Lagrangian in equation (4.49) yields the equation of motion (4.50).

4.4 A) Solve the differential equation (4.56). B) Check your answer by substituting your solution back into the equation.

4.5 Now put your solution from problem 4.4 into the Friedmann equation (4.53) and solve for $R(t)$.

4.6 A) If $(\frac{\dot{R}}{R})^2 \propto \frac{1}{R^m}$, show that $R \propto t^{2/m}$. B) If $H^2 \equiv (\frac{\dot{R}}{R})^2 = constant$ show that $R \propto e^{Ht}$.

4.7 If $(\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3} V_o$, show that $R = R_o e^{\sqrt{\frac{8\pi G}{3} V_o}(t-t_o)}$.

4.8 Show that inflationary solutions are characterized by $\ddot{R} > 0$ and non-inflationary expansions by $\ddot{R} < 0$

4.9 Barrow's model is $V(\phi) \equiv \beta e^{-\lambda\phi}$. Check that $\phi(t) = \sqrt{2A} \ln t$ is a soluttion and evaluate the constants β and λ in terms of A.

4.10 Prove equations (4.103)-(4.108).

Chapter 5

EINSTEIN FIELD EQUATIONS

The Einstein's field equations are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{5.1}$$

which are a set of 16 coupled equations which will give $g_{\mu\nu}$ (buried inside $G_{\mu\nu}$) gives $T_{\mu\nu}$. Actually there are only 10 independent equations because of the symmetry $g_{\mu\nu} = g_{\nu\mu}$.

In principle our job is easy. Just write down $T_{\mu\nu}$ and solve for $g_{\mu\nu}$ which specifies the metric. Then we can calculate the paths of light rays, the orbits of planets, etc. In practice the solution of the Einstein field equations are exceedingly difficult and only a few exact solutions are known.

Two excellent reference for this section are Chapter 7 of the book by Lawden [**?**] and Chapter 2 of the book by Islam [13].

In actual practice, the way one usually solves the Einstein's equations is to specify a metric in general terms which contains unknown coefficients. This metric is substituted into the Einstein equations and one solves for the unknown coefficients.

Thus we need to learn how to derive the metric for the spaces under consideration. Let us learn how to derive the Friedmann-Robertson-Walker (FRW) metric which is the metric appropriate to a *homogeneous* and *isotropic* universe but where *size can change with time*.

5.1 Preview of Riemannian Geometry

5.1.1 Polar Coordinate

A general N-dimensional Riemannian space, denoted by $\mathcal{R}_{\mathcal{N}}$, is one in which the distance ds between two neighboring points can be written (Pg. 88 [**?**])

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{5.2}$$

If coordinates can be found such that

$$ds^2 = dx^\mu dx^\nu \tag{5.3}$$

over the *whole* space then the space is said to be Euclidean and is denoted by $\mathcal{E}_N$. Clearly $\mathcal{E}_N$ is a special case of $\mathcal{R}_N$.

Below we shall restrict our discussion to the *spatial* part of the metric denoted as

$$d\ell^2 = h_{ij} dx^i dx^j. \tag{5.4}$$

We shall very often have (eg. FRW and special relativity metrics)

$$ds^2 = c^2 dt^2 - d\ell^2 \tag{5.5}$$

so that

$$h_{ij} = -g_{ij}. \tag{5.6}$$

Consider the two dimensional space where

$$d\ell^2 = dr^2 + r^2 d\theta^2. \tag{5.7}$$

Here $h_{11} = 1$, $h_{12} = h_{21} = 0$, $h_{22} = r^2$. The space *looks* like $\mathcal{R}_2$ but actually it is $\mathcal{E}_2$ because we can find coordinate such that (5.3) is true. These coordinate are the two-dimensional plane polar coordinates

$$x = r\cos\theta \tag{5.8}$$
$$y = r\sin\theta \tag{5.9}$$

in which

$$d\ell^2 = dx^2 + dy^2, \tag{5.10}$$

with $g_{11} = g_{22}$=1 and $g_{12} = g_{21} = 0$ or $g_{\mu\nu} = \delta_{\mu\nu}$.

Recall that for 3-dimensional spherical polar coordinates

$$x = r\sin\theta\cos\phi \tag{5.11}$$
$$y = r\sin\theta\sin\phi \tag{5.12}$$
$$z = r\cos\theta \tag{5.13}$$

and the increments of length $d\ell_r$, $d\ell_\theta$, $d\ell_\phi$ is the e_r, e_θ and e_ϕ directions respectively are

$$d\ell_r = dr \tag{5.14}$$
$$d\ell_\theta = rd\theta \tag{5.15}$$
$$d\ell_\phi = r\sin\theta d\phi. \tag{5.16}$$

Thus the surfaces of a sphere is an example of a space which is $\mathcal{R}_2$ and cannot be reduced to $\mathcal{E}_2$. On the surface of the sphere the distance between two points is

$$d\ell^2 = d\ell_\theta^2 + d\ell_\phi^2 \tag{5.17}$$
$$= r^2 d\theta^2 + r^2\sin^2\theta d\phi^2 \tag{5.18}$$

where $h_{11} = r^2$, $h_{12} = h_{21} = 0$, $h_{22} = r^2\sin^2\theta$. For this surface it is *not* possible to find x, y such that $ds^2 = dx^2 + dy^2$ and therefore the surface of a sphere is not $\mathcal{E}_\in$ but rather a genuine $\mathcal{R}_\in$ space.

5.1.2 Volumes and Change of Coordinates

The *measure* is the volume element in some set of coordinates. There are 3 ways to calculate the measure.

1) *Infinitesimal Length Method* is one in which one identifies the infintesimal increments of length and simply multiplies them together to get the volume element. In Cartesian Coordinates we have $d\ell_x = dx$, $d\ell_y = dy$, and $d\ell_z = dz$ to give

$$dV = d\ell_x d\ell_y d\ell_z = dxdydz. \tag{5.19}$$

In 2-d plane polar coordinates $d\ell_r = dr$ and $d\ell_\theta = rd\theta$ to give

$$dV = d\ell_r d\ell_\theta = rdrd\theta \tag{5.20}$$

(actually this "volume" is an area). In 3-D spherical polar coordinates $d\ell_r = dr$, $d\ell_\theta = d\theta$ and $d\ell_\phi = r\sin d\phi$ to give

$$dV = d\ell_r d\ell_\theta d\ell_\phi = r^2\sin\theta drd\theta d\phi. \tag{5.21}$$

2) *Jacobian Method* is the one usually mentioned in introductory calculus books ([**?**] Pg.746) for changing variables. Suppose $x = x(u,v)$ and $y = y(u,v)$ then

$$\int\int f(x,y)dxdy = \int\int f[x(u,v),y(u,v)] \mid J(u,v) \mid dudv \tag{5.22}$$

where $\mid J(u,v) \mid$ is the modulus of the Jacobian defined as

$$J(u,v) \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \tag{5.23}$$

For 3-dimensions with $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$ we have

$$J(u,v,w) \equiv \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \tag{5.24}$$

For Cartesian coordinates obviously $J(x,y,z) = 1$. For plane polar coordinates ($x = r\cos\theta$, $y = r\sin\theta$)

$$J(r,\theta) = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r \tag{5.25}$$

and for spherical polar coordinates ($x = r\sin\theta\cos\phi$, $y = r\sin\theta\cos\phi$, $z = r\cos\theta$)

$$J(r,\theta,\phi) = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} = r^2\sin\theta \tag{5.26}$$

The volume element in 2-d is

$$dV = \mid J(u,v) \mid dudv \tag{5.27}$$

and in 3-d it is

$$dV = \mid J(u,v,w) \mid dudvdw \tag{5.28}$$

which then reproduce equations (5.19), (5.139) and (5.140) for Cartesian, plane polar and spherical polar coordinates.

3) *Metric Tensor Method* is what we prefer in general relativity. Here ([21], Pg.33)

$$\boxed{dV = \sqrt{h} du dv dw} \tag{5.29}$$

where h is the *determinant* of the spatial metric tensor. Thus

$$\boxed{\mid J \mid = \sqrt{h}} \tag{5.30}$$

For a plane polar coordinates $ds^2 = dr^2 + r^2 d\theta^2 \equiv g_{ij} dx^i dx^j$ so that

$$h = \begin{vmatrix} 1 & 0 \\ 0 & r^2 \end{vmatrix} = r^2 \tag{5.31}$$

giving $\sqrt{h} = r$ so that $dV = \sqrt{h} dr d\theta$ in agreement with (5.139). For spherical polar coordinates $d\ell^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \equiv h_{ij} dx^i dx^j$ giving

$$h = \begin{vmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{vmatrix} = r^4 \sin^2\theta \tag{5.32}$$

giving $\sqrt{h} = r^2 \sin\theta$ so that $dV = \sqrt{h} dr d\theta d\phi = r^2 \sin\theta dr d\theta d\phi$ in agreement with (5.140).

Thus *define*

$$\begin{aligned} d^2x &\equiv dudv \\ d^3x &\equiv dudvdw \\ d^4x &\equiv dudvdwdt \end{aligned} \tag{5.33}$$

so that the measure is

$$\boxed{dV = \sqrt{h} d^2x \text{ or } \sqrt{h} d^3x} \tag{5.34}$$

depending on the number of dimensions. It is *important* to remember that d^3x or d^4x om this notation is not $d\ell_1 d\ell_2 d\ell_3$ or $d\ell_1 d\ell_2 d\ell_3 d\ell_4$ but simply *only* the coordinates. For example in spherical polar coordinates

$$d^3x \equiv dr d\theta d\phi \tag{5.35}$$

or with time

$$d^4x \equiv dr d\theta d\phi dt. \tag{5.36}$$

The measure is volume is obtained with $dV = \sqrt{h} d^3x$ or $\sqrt{-g} d^4x$ (because for 4-d we use $g_{\mu\nu}$ and $h_{ij} = -g_{ij}$ and $h = -g$). Thus in general

$$dV \neq d^3x$$

or

$$dV \neq d^4x$$

in contrast to conventions used, say in undergraduate physics books.

5.1.3 Differential Geometry

A Good reference for differential geometry is the book by Lipschutz [**?**]. Introductory material is discussed by Purcell and Varberg [**?**] (Pg.625-634, 285-290).

Previously we wrote the definition of a circle as $x^2 + y^2 = r^2$ which could be written generally as $y = y(x)$. The same equation can be expressed *parametrically* in terms of the *parameter* θ as $x = r\cos\theta$ and $y = r\sin\theta$ or generally as $x = x(\theta)$ and $y = y(\theta)$. For many curves the form $y = y(x)$ can be clumsy and nowadays mathematicians always prefer the parametric representation. (See [**?**] Pg.570). Another way to write our equation for the circle (radius=1) is

$$\mathbf{x} = \cos\theta\hat{e}_1 + \sin\theta\hat{e}_2 \tag{5.37}$$

where $\hat{e}_1$ and $\hat{e}_2$ are basis vectors in $\mathcal{E}_2$.

Thus a general *curve* (Chapter 3 of [**?**]) is expressed as

$$\mathbf{x} = \mathbf{x}(t) \tag{5.38}$$

where t is the parameter. If the basis is chosen to be $\mathcal{E}_2$ then $\mathbf{x} = \mathbf{x}(t)$ is equivalent to two scalar equations $x_1 = x_1(t)$ and $x_2 = x_2(t)$. Thus a curve can be specified in *any* number of dimensions. For our circle above we have $x_1 = x_1(\theta) = \cos\theta$ and $x_2 = x_2 = \sin\theta$.

A general *surface* (Chapter 8 of [**?**]) is expressed as

$$\mathbf{x} = \mathbf{x}(u, v). \tag{5.39}$$

If, for example, the basis in $\mathcal{E}_3$ then

$$\mathbf{x}(u, v) = x_1(u, v)\hat{e}_1 + x_2(u, v)\hat{e}_2 + x_3(u, v)\hat{e}_3. \tag{5.40}$$

Good references of the next 3 sections are the books by Kolb and Turner ([21], Pg.31-35) and Landau ([**?**], Pg.177-181) and Ohanian and Ruffini ([8], Pg546-552) and Chow ([28], Pg.197-200) and the article by Kung [**?**].

5.1.4 1-dimesional Curve

Let us first consider the circle, often called the one sphere denoted by S^1. Recall that for a circle the radius R, the proper way to express it is in terms of 1-dimensional parameter θ as

$$\mathbf{x}(\theta) = R(\cos\theta\hat{e}_1 + \sin\theta\hat{e}_2). \tag{5.41}$$

However in introductory books one always *introduces a ficticous extra dimension and embeds the 1-d curve in a 2-d Euclidean space* via

$$x^2 + y^2 = R^2 \tag{5.42}$$

which we recognize as the equation for a circle. Remember though this equation is really overkill. *It is a 2-d equation for a 1-d curve*! The 1-parameter equation (5.41) is much better. We can also write

$$x_1^2 + x_2^2 = R^2. \tag{5.43}$$

The element of length in the 2-d Euclidean space is

$$d\ell^2 = dx_1^2 + dx_2^2. \tag{5.44}$$

In an ordinary 2-d Euclidean space x and y (or x_1 and x_2) are free to vary independently and this is how the whole 2-d space get covered. Equation (5.44) is true in general. However the reason that (5.42) or (5.43) describes a circle is because it *constrains* the value of y in terms of x. This constraint (5.42) picks our only those points in $\mathcal{E}_2$ which give the circle.

Equation (5.44) covers all of $\mathcal{E}_2$. We can constrain it for the circle by reducing the two parameters x_1 and x_2 to only one parameter. Thus we will have dℓ for the circle. We do this using the 2-d constraint (5.43) and writing $y = \sqrt{R^2 - x^2}$ and $dy = \frac{-x}{\sqrt{R^2-x^2}}dx$ so that

$$dy^2 = \frac{x^2}{R^2 - x^2}dx^2. \tag{5.45}$$

Note that $dy^2 \equiv (dy)^2$ and $dy^2 \neq d(y^2)$. Thus (5.44) becomes

$$\begin{aligned} d\ell^2 &= dx^2 + \frac{x^2}{R^2 - x^2}dx^2 \\ &= \frac{R^2}{R^2 - x^2}dx^2. \end{aligned} \tag{5.46}$$

This can also be written in terms of the *dimensionless* coordinate

$$r \equiv \frac{x}{R} \tag{5.47}$$

to give

$$d\ell^2 = R^2 \frac{dr^2}{1-r^2} \tag{5.48}$$

where R is the radius of the space (the circle).

Another convenient coordinate system for the circle use the angle θ from plane polar coordinates specified via

$$\begin{aligned} x &= R\cos\theta \\ y &= R\sin\theta. \end{aligned} \tag{5.49}$$

Identifying the increments of length $d\ell_R$ and $d\ell_\theta$ in the $\hat{e}_R$ and $\hat{e}_\theta$ directions as

$$\begin{aligned} d\ell_R &= dR \\ d\ell_\theta &= Rd\theta \end{aligned} \tag{5.50}$$

then

$$\begin{aligned} d\ell^2 &= d\ell_R^2 + d\ell_\theta^2 \\ &= dR^2 + R^2 d\theta^2 \end{aligned} \tag{5.51}$$

which gives the distance $d\ell$ in the *2-d* space. To restrict ourselves to the rim of the circle (curved 1-d space) we fix $d\ell_R = dR = 0$ and get

$$d\ell^2 = R^2 d\theta^2 \tag{5.52}$$

which makes it obvious that the space is the one sphere (circle) of radius R. Using simple trigonometry one can show that (5.52) is the *same* as (5.46). (**Do Problem 5.1**)

Using $d\ell^2 \equiv h_{ij}dx^i dx^j$ we evidently have

$$h_{ij} = (R^2) \tag{5.53}$$

which is a 1-dimensional "motion". The determinant is obviously $h = R^2$ giving $\sqrt{h} = R$. This allows us to calculate the volume (we are calling the length a geological volume) as

$$V = \int \sqrt{h} d^1x = \int_0^{2\pi} R d\theta = 2\pi R \tag{5.54}$$

The 1-d curve that we described above is the circle or one sphere denoted S^1. However there are *three* 1-d spaces which are homogeneous and isotropic. There are i) the flat x line (R^1), ii) the positively curved one sphere (S^1) derived above and iii) the negatively curved hyperbolic curve (H^1). [21]

The formulas for a space of constant negative curvature [21] can be obtained with the replacement

$$R \to iR \tag{5.55}$$

to yield

$$\begin{aligned} d\ell^2 &= \frac{-R^2}{-R^2 - x^2} dx^2 \\ &= \frac{R^2}{R^2 + x^2} dx^2 \end{aligned} \tag{5.56}$$

or using $r \equiv \frac{x}{R}$

$$d\ell^2 = R^2 \frac{dr^2}{1 + r^2}. \tag{5.57}$$

These results are also obtained by embedding in Minkowski space (**do Problem 5.2**).

The line element for a space of zero curvature is obviously just

$$d\ell^2 = dx^2 \tag{5.58}$$

or using $r \equiv \frac{x}{R}$

$$d\ell^2 = R^2 dr^2. \tag{5.59}$$

These formulas are obtained from S^1 or H^1 by letting $R \to \infty$.

We can collect our results for R^1, S^1 and H^1 into a single formula

$$d\ell^2 = \frac{R^2}{R^2 - kx^2} dx^2 \tag{5.60}$$

or using $r \equiv \frac{x}{R}$

$$\boxed{d\ell^2 = R^2 \frac{dr^2}{1 - kR^2}} \tag{5.61}$$

where $k = 0, +1, -1$ for flat, closed and open curves respectively. (i.e. for R^1, S^1 and H^1 respectively)

5.1.5 2-dimensional Surface

Kolb and Turner [21] analyze this problem very nicely. Other references are listed following equation (5.40).

A surface is represented by two parameters u, v and expressed as $\mathbf{x} = \mathbf{x}(u, v)$ as mentioned previously. However we shall introduce a surface ficticious coordinate (three parameters) and embedding the surface in $\mathcal{E}_3$. Thus with 3 parameters the equation for the two sphere is

$$x^2 + y^2 + z^2 + R^2 \tag{5.62}$$

which we recognize as the equation for a sphere used in introductory books. However this equation is overkill. It is a 3-d equation for a 2-d surface. We can also write

$$x_1^2 + x_2^2 + x_3^2 = R^2. \tag{5.63}$$

Any 3-d Euclidean space $\mathcal{E}_3$ has length element

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 \tag{5.64}$$

which under normal circumstances would map out the whole 3-d volume. However (5.63) *restricts* x_3 according to

$$x_3^2 = R^2 - x_1^2 - x_2^2. \tag{5.65}$$

Writing

$$dx_3 = \frac{\partial x_3}{\partial x_1} dx_1 + \frac{\partial x_3}{\partial x_2} dx_2 \tag{5.66}$$

and with $\frac{\partial x_3}{\partial x_1} = \frac{-x_1}{\sqrt{R^2 - x_1^2 - x_2^2}}$ we have

$$dx_3 = -\frac{x_1 dx_1 + x_2 dx_2}{\sqrt{R^2 - x_1^2 - x_2^2}} \tag{5.67}$$

or

$$dx_3^2 = \frac{(x_1 dx_1 + x_2 dx_2)^2}{R^2 - x_1^2 - x_2^2} \tag{5.68}$$

to give ([21], Pg.32)

$$d\ell^2 = dx_1^2 + dx_2^2 + \frac{x_1 dx_1 + x_2 dx_2)^2}{R^2 - x_1^2 - x_2^2} \tag{5.69}$$

which is re-written as

$$d\ell^2 = \frac{1}{R^2 - x_1^2 - x_2^2}[(R^2-x_2^2)dx_1^2+(R^2-x_1^2)dx_2^2+x_1x_2dx_1dx_2+x_2x_1dx_2dx_1]. \tag{5.70}$$

Let us introduce plane polar coordinates in the x_3 plane as

$$x_1 = r' \cos\theta \qquad x_2 = r' \sin\theta. \tag{5.71}$$

These coordinates are shown *very* clearly in Fig 2.1, Pg.32 of Kolb and Turner [21]. Thus

$$\begin{aligned} \theta &: 0 \to 2\pi \\ r' &: 0 \to R. \end{aligned} \tag{5.72}$$

Using

$$dx_i = \frac{\partial x_i}{\partial x^1}dr' + \frac{\partial x_i}{\partial \theta}d\theta \tag{5.73}$$

then (5.69) and (5.70) become

$$d\ell^2 = \frac{R^2}{R^2 - r'^2}dr'^2 + r'^2 d\theta^2. \tag{5.74}$$

This can also be written in terms of the *dimensionless* coordinate

$$r \equiv \frac{r'}{R} \tag{5.75}$$

to give

$$d\ell^2 = R^2[\frac{dr^2}{1-r^2} + r^2 d\theta^2] \tag{5.76}$$

where

$$r : 0 \to 1. \tag{5.77}$$

Another convenient coordinate system for the two sphere uses angles θ and ϕ from spherical polar coordinates specified via

$$x = R\sin\theta\cos\phi \tag{5.78}$$

$$y = R\sin\theta\sin\phi \tag{5.79}$$

$$z = R\cos\theta \tag{5.80}$$

and substituting into (5.64) directly yields

$$d\ell^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.81}$$

Equation (5.81) is *alternatively* obtained by identifying the increments of length $d\ell_R$, $d\ell_\theta$, $d\ell_\phi$ in the $\hat{e}_R$, $\hat{e}_\theta$ and $\hat{e}_\phi$ directions as

$$d\ell_R = dR \tag{5.82}$$
$$d\ell_\theta = Rd\theta \tag{5.83}$$
$$d\ell_\phi = R\sin\theta d\phi \tag{5.84}$$

then

$$\begin{aligned} d\ell^2 &= d\ell_R^2 + d\ell_\theta^2 + d\ell_\phi^2 \\ &= dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned} \tag{5.85}$$

gives the distance ds in the *3-d* space. To restrict ourselves to the surface of the sphere (curved 2-d space) we find $d\ell_R = dR = 0$ and get

$$d\ell^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5.86}$$

in agreement with (5.81).

Using $d\ell^2 \equiv h_{ij}dx^i dx^j$ we evidently have for S^2

$$h_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2\sin^2\theta \end{pmatrix} \tag{5.87}$$

The determinant is obviously

$$h = R^4\sin^2\theta \tag{5.88}$$

giving $\sqrt{h} = R^2\sin\theta$. The volume (we are calling the surface area a generalized volume) is

$$V = \int\sqrt{h}d^2x = \int\sqrt{h}d\theta d\phi + R^2\int_0^\pi \sin\ theta\int_0^{2\pi} d\phi = 4\pi R^2 \tag{5.89}$$

Actually there are *three* 2-d spaces which are homogenous and isotropic. There are 1) the flat $x - y$ plane (R^2), ii) the positively curved two sphere (S^2) and iii) the negatively curved two hyperbola (H^2).

As before we can obtain the formula for H^2 with the replacement

$$R \to iR \tag{5.90}$$

to yield

$$d\ell^2 = R^2[\frac{dr^2}{1+r^2} + r^2d\theta^2]. \tag{5.91}$$

This result is also obtained by embedding in Minkowski space. The metric corresponding to (5.81) is ([21], Pg.34, equation 2.17)

$$d\ell^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{5.92}$$

We can collect our results for R^2, S^2 and H^2 into a single formula

$$d\ell^2 = R^2(\frac{dr^2}{1-kr^2} + r^2 d\theta^2) \tag{5.93}$$

where k=0, +1, -1 for flat (R^2), closed (S^2) and open (H^2) surfaces respectively.

The volume can be alternatively calculated using $d\ell^2 = h_{ij}dx^i dx^j$ in (5.93) we have

$$h_{ij} = \begin{pmatrix} \frac{R^2}{1-kr^2} & 0 \\ 0 & R^2 r^2 \end{pmatrix} \tag{5.94}$$

giving the determinant

$$h = \frac{R^2 r^2}{1-kr^2} \tag{5.95}$$

or $\sqrt{h} = \frac{R^2 r}{\sqrt{1-kr^2}}$. The volume is

$$\begin{aligned} V &= \int \sqrt{h} d^2x = \int \sqrt{h} = R^2 \int_0^0 \frac{rdr}{\sqrt{1-kr^2}} \int_0^{2\pi} d\theta && (5.96) \\ &= 2\pi R^2 \int_0^0 \frac{rdr}{\sqrt{1-kr^2}}. && (5.97) \end{aligned}$$

The limits of integration $\int_0^0 dr$ can be clearly seen from Fig 2.1 of Kolb and Turner [21], Pg.32. What this really means is

$$\int_0^0 dr \equiv 2\int_0^R dr \tag{5.98}$$

where $r = 0$ at $\theta = 0$ and $r = R$ at $\theta = \frac{\pi}{2}$ and $r = 0$ again at $\theta = \pi$. (See bottom Pg.179 and top pg.180 of the book by Lawden [**?**] for more explanation)

The integral and its limits are more clearly done with the substitution

$$\sqrt{k}r \equiv \sin\chi \tag{5.99}$$

when

$$\chi \;:\; 0 \;\rightarrow\; \pi. \tag{5.100}$$

Thus $\int_0^\pi d\chi$ is equivalent to $\int_0^0 dr = 2\int_0^R dr$. Thus the volume in equation (5.97) becomes

$$V = 2\pi R^2 \int_0^\pi \frac{1}{\sqrt{k}}\frac{1}{\sqrt{k}} \sin\chi d\chi \tag{5.101}$$

giving

$$V = \frac{4\pi R^2}{k}. \tag{5.102}$$

Thus for $k = +1$ we have $V = 4\pi R^2$ as before. For $k = 0$ we have $V = \infty$ and for $k = -1$ we need to do the integral again. We would find $V = \infty$ for $k = -1$.

5.1.6 3-dimensional Hypersurface

Proceeding upwards in our number of dimensions we might inquire a "curved volume". But the curvature can really only be imagined with respect to embedding in a 4-dimensional Euclidean space $\mathcal{E}_4$. We call the 4-d space as a *hypersurface*.

Our 4-d Euclidean space (into which we will embed the hypersurface) has length element

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 \tag{5.103}$$

(it *is* $d\ell^2$ and *not* ds^2).

A hypersurface is represented by three parameters u, v, w and is expressed as $\mathbf{x} = \mathbf{x}(u,v,w)$. Introduce an extra ficticious coordinate for the three sphere S^3 as

$$\begin{aligned} x^2 + y^2 + z^2 + w^2 &= R^2 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 &= R^2 \end{aligned} \tag{5.104}$$

which restricts x_4 as

$$x_4^2 = R^2 - x_1^2 - x_2^2 - x_3^2. \tag{5.105}$$

Writing

$$dx_4 = \frac{\partial x_4}{\partial x_1}dx_1 + \frac{\partial x_4}{\partial x_2}dx_2 + \frac{\partial x_4}{\partial x_3}dx_3 \tag{5.106}$$

and with $\frac{\partial x_4}{\partial x_1} = \frac{-x_1}{\sqrt{R^2-x_1^2-x_2^2-x_3^2}}$ etc. we have

$$dx_4 = -\frac{x_1 dx_1 + x_2 dx_2 + x_3 dx_3}{\sqrt{R^2 - x_1^2 - x_2^2 - x_3^2}} \tag{5.107}$$

or

$$dx_4^2 = \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - x_1^2 - x_2^2 - x_3^2} \tag{5.108}$$

to give ([21], Pg.34)

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - x_1^2 - x_2^2 - x_3^2}. \tag{5.109}$$

Let us introduce spherical polar coordinates in the x_4 hyperplane as

$$x_1 = r' \sin\theta \cos\phi \tag{5.110}$$
$$x_2 = r' \sin\theta \sin\phi \tag{5.111}$$
$$x_3 = r' \cos\theta \tag{5.112}$$

where

$$\theta : 0 \to \pi \tag{5.113}$$
$$\phi : 0 \to 2\pi \tag{5.114}$$
$$r' : 0 \to R. \tag{5.115}$$

Using

$$dx_i = \frac{\partial x_i}{\partial r'} dr'. + \frac{\partial x_i}{\partial \theta} d\theta + \frac{\partial x_i}{\partial \phi} d\phi \tag{5.116}$$

then (5.109) becomes

$$d\ell^2 = \frac{R^2}{R^2 - r'^2} dr'^2 + r'^2 d\theta^2 + r'^2 \sin^2\theta d\phi^2 \tag{5.117}$$

Introducing the dimensionless coordinate

$$r \equiv \frac{r'}{R} \tag{5.118}$$

gives

$$d\ell^2 = R^2 [\frac{dr^2}{1 - r^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] \tag{5.119}$$

where

$$r \; : \; 0 \; \rightarrow \; 1. \tag{5.120}$$

Another convenient coordinate system for the three sphere uses angles χ, θ, ϕ from 4-dimensional hyperspherical polar coordinates specified via

$$\begin{aligned} x &= R \sin\chi \sin\theta \cos\phi & (5.121) \\ y &= R \sin\chi \sin\theta \sin\phi & (5.122) \\ z &= R \sin\chi \cos\theta & (5.123) \\ w &= R \cos\chi. & (5.124) \end{aligned}$$

Substituting into (5.119) directly yields

$$d\ell^2 = R^2[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{5.125}$$

Using $d\ell^2 = h_{ij}dx^i dx^j$ we have

$$h_{ij} = \begin{pmatrix} R^2 & 0 & 0 \\ 0 & R^2\sin^2\chi & 0 \\ 0 & 0 & R^2\sin^2\chi\sin^2\theta \end{pmatrix} \tag{5.126}$$

The determinant is

$$h = R^6 \sin^4\chi \sin^2\theta \tag{5.127}$$

giving $\sqrt{h} = R^3 \sin^2\chi \sin\theta$. The volume is

$$\begin{aligned} V &= \int \sqrt{h} d^3x = \int \sqrt{h} d\chi d\theta d\phi \\ &= R^3 \int_0^\pi \sin^2\chi d\chi \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi & (5.128) \end{aligned}$$

where the limits $\int_0^\pi d\chi$ are the same as in the previous section. Thus

$$V = 4\pi R^3 \int_0^\pi \sin^2\chi d\chi = 4\pi R^3 [\frac{\chi}{2} - \frac{\sin 2\chi}{4}]_0^\pi \tag{5.129}$$

giving

$$V = 2\pi^2 R^3 \tag{5.130}$$

for the volume of our hypersphere. Compare this to the volume of a Euclidean sphere $\frac{4}{3}\pi R^3$.

For a flat, open and closed hyperspheres the metric is

$$d\ell^2 = R^2[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2] \tag{5.131}$$

The volume can be calculated alternatively. Using $d\ell = h_{ij}dx_i dx_j$ in (5.131) we have

$$h_{ij} = \begin{pmatrix} \frac{R^2}{1-kr^2} & 0 & 0 \\ 0 & R^2r^2 & 0 \\ 0 & 0 & R^2r^2\sin^2\theta \end{pmatrix} \tag{5.132}$$

giving the determinant $h = \frac{R^6 r^4 \sin^2\theta}{1-kr^2}$. The volume is

$$\begin{aligned} V = \int \sqrt{h} d^3x &= \int \sqrt{h} dr d\theta d\phi \\ &= R^3 \int_0^0 \frac{r^2 dr}{\sqrt{1-kr^2}} \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi \\ &= 4\pi R^3 \int_0^0 \frac{r^2 dr}{\sqrt{1-kr^2}}. \end{aligned} \tag{5.133}$$

The limits of integration are the same as discussed in equation (5.98).

Using the substitution

$$\sqrt{k}r \equiv \sin\chi' \tag{5.134}$$

with

$$\chi' \; : \; 0 \; \rightarrow \; \pi \tag{5.135}$$

becuase $\chi \; : \; 0 \; \rightarrow \; \pi$. Thus the volume is

$$V = 4\pi R^3 \int_0^\pi \frac{1}{k^{3/2}} \sin^2\chi d\chi \tag{5.136}$$

giving

$$V = \frac{2\pi^2 R^3}{k^{3/2}}. \tag{5.137}$$

For $k = +1$ this agrees with our result before.

5.2 Friedmann-Robertson-Walker Metric

The metric of Special Relativity is

$$ds^2 = c^2 dt - (dx^2 dy^2 dz^2). \tag{5.138}$$

Clearly the spatial part is a 3-d Euclidean flat space. We have seen that the spatial metric for a homogeneous, isotropic curved space with a size $R(t)$ that can change in time is

$$ds^2 = R^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)]. \tag{5.139}$$

Replacing the spatial part of the special relativity metric with (5.139) we have the Friedmann-Robertson-Walker (FRW) metric [13]

$$ds^2 = c^2dt^2 - R^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)] \tag{5.140}$$

where $R(t)$ is called the scale factor and the constant k can be $0, \pm 1$ depending on the curvature. *This can also be derived with the use of Killing vectors* [13].

Writing $ds^2 \equiv g_{\mu\nu}dx^\mu dx^\nu$ and identifying

$$x^0 = ct \tag{5.141}$$

$$x^1 = r \tag{5.142}$$

$$x^2 = \theta \tag{5.143}$$

$$x^3 = \phi \tag{5.144}$$

we have

$$g_{00} = 1 \tag{5.145}$$

$$g_{11} = \frac{-R^2}{1-kr^2} \tag{5.146}$$

$$g_{22} = -R^2r^2 \tag{5.147}$$

$$g_{33} = -R^2r^2\sin^2\theta \tag{5.148}$$

Defining the determinant

$$g \equiv det g_{\mu\nu} = g_{00}g_{11}g_{22}g_{33} \tag{5.149}$$

$$= -\frac{R^6r^4\sin^2\theta}{1-kr^2} \tag{5.150}$$

(Note that this is *not* $g = det g^{\mu\nu}$.) Thus

$$\sqrt{-g} = \frac{R^3r^2\sin\theta}{\sqrt{1-kr^2}}. \tag{5.151}$$

If $g_{\mu\nu}$ is represented by a matrix $[g_{\mu\nu}]$, then we found previously that $g^{\mu\nu}$ is just the inverse of this metric namely $[g_{\mu\nu}]^{-1}$. For a diagonal matrix (which we have for the FRW metric) each matrix element is simply given by $g^{\mu\nu} = \frac{1}{g_{\mu\nu}}$. Thus it's easy to get

$$g^{00} = 1 \tag{5.152}$$

$$g^{11} = -\frac{(1-kr^2)}{R^2} \tag{5.153}$$

$$g^{22} = \frac{-1}{R^2 r^2} \tag{5.154}$$

$$g^{33} = \frac{-1}{R^2 r^2 \sin^2\theta}. \tag{5.155}$$

5.2.1 Christoffel Symbols

We now calculate the Christoffel symbols using equation (3.69). Fortunately we need not calculate all of them. We can use the symmetry $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta}$ to shorten the job. We have

$$\begin{aligned}\Gamma^\alpha_{\beta\gamma} &\equiv \frac{1}{2}g^{\alpha\epsilon}(g_{\epsilon\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\epsilon}) = \Gamma^\alpha_{\gamma\beta} \\ &= \frac{1}{2}g^{\alpha\alpha}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha})\end{aligned} \tag{5.156}$$

which follows because $g^{\alpha\epsilon} = 0$ unless $\epsilon = \alpha$. ($g^{\mu\nu}$ is a diagonal matrix for the FRW metric.) The only *non-zero* Christoffel symbols are the following:

$$\Gamma^0_{11} = \frac{1}{2}g^{00}(g_{01,1} + g_{01,1} - g_{11,0}) = -\frac{1}{2}g_{11,0}$$

because $g_{01} = 0$ and $g^{00} = 1$. This becomes (let's now set $c \equiv 1$)

$$\begin{aligned}\Gamma^0_{11} &= -\frac{1}{2}g_{11,0} = -\frac{1}{2}\frac{\partial}{\partial t}(\frac{-R^2}{1-kr^2}) \\ &= \frac{1}{2}\frac{1}{1-kr^2}\frac{\partial R^2}{\partial t} = \frac{2R\dot{R}}{2(1-kr^2)} = \frac{R\dot{R}}{1-kr^2}\end{aligned} \tag{5.157}$$

because $r \neq r(t)$ and $R = R(t)$. Proceeding

$$\Gamma^0_{22} = -\frac{1}{2}g_{22,0} = -\frac{1}{2}\frac{\partial}{\partial t}(-R^2 r^2) = r^2 R\dot{R} \tag{5.158}$$

$$\Gamma^0_{33} = r^2\sin^2\theta R\dot{R} \tag{5.159}$$

$$\Gamma^1_{11} = \frac{kr}{1-kr^2} \tag{5.160}$$

$$\Gamma^1_{22} = -r(1-kr^2) \tag{5.161}$$

$$\Gamma^1_{33} = -r(1-kr^2)\sin^2\theta \tag{5.162}$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r} \tag{5.163}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta \tag{5.164}$$

$$\Gamma^3_{23} = \cot\theta \tag{5.165}$$

$$\Gamma^1_{01} = \Gamma^2_{02} = \Gamma^3_{03} = \frac{\dot{R}}{R} \tag{5.166}$$

(**do Problems 5.2 and 5.3**)

5.2.2 Ricci Tensor

Using equation (**??**) we can now calculate the Ricci tensor. For the FRW metric it turns out that $R_{\mu\nu} = 0$ for $\mu \neq \nu$, so that the non-zero components are R_{00}, R_{11}, R_{22}, R_{33}. Proceeding we have

$$R_{00} = \frac{1}{\sqrt{-g}}(\Gamma^\epsilon_{00}\sqrt{-g}), -(\ln\sqrt{-g})_{,00} - \Gamma^\epsilon_{0\theta}\Gamma^\theta_{0\epsilon}$$

but $\Gamma^\epsilon_{00} = 0$ giving

$$R_{00} = -(\ln\sqrt{-g})_{,00} - \Gamma^0_{0\theta}\Gamma^\theta_{00} - \Gamma^1_{0\theta}\Gamma^\theta_{01} - \Gamma^2_{0\theta}\Gamma^\theta_{02} - \Gamma^3_{0\theta}\Gamma^\theta_{03}$$

when we have performed the sum over ϵ. The term $\Gamma^\theta_{0\theta} = 0$. In the last three terms we have $\Gamma^\alpha_{0\theta}$ where $\alpha = 1, 2, 3$. Now $\Gamma^\alpha_{0\theta} = 0$ for $\theta \neq \alpha$, so that we must have $\theta = 1, 2, 3$ in the third, forth and fifth terms respectively. Also the second term contains $\Gamma^0_{0\theta}$ which is always 0. Thus

$$\begin{aligned}
R_{00} &= -(\ln\sqrt{-g})_{,00} - \Gamma^1_{10}\Gamma^1_{01} - \Gamma^2_{02}\Gamma^2_{02} - \Gamma^3_{03}\Gamma^3_{03} \\
&= -(\ln\sqrt{-g})_{,00} - (\Gamma^1_{01})^2 - (\Gamma^2_{02})^2 - (\Gamma^3_{03})^2 \\
&= -(\ln\sqrt{-g})_{,00} - 3(\frac{\dot{R}}{R})^2
\end{aligned}$$

Now

$$(\sqrt{-g})_{,0}\frac{\partial\sqrt{-g}}{\partial x'} = \frac{\partial\sqrt{-g}}{\partial t} = \frac{r^2\sin\theta}{\sqrt{1-kr^2}}\frac{\partial R^3}{\partial t} = \frac{r^2\sin\theta}{\sqrt{1-kr^2}}3R^2\dot{R}$$

and

$$(\ln\sqrt{-g})_{,\mu} \equiv \frac{\partial\ln\sqrt{-g}}{\partial x^\mu} = \frac{\partial\ln\sqrt{-g}}{\partial\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\mu} = \frac{1}{\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^\mu}$$

so that

$$(\ln \sqrt{-g})_{,0} = \frac{1}{-\sqrt{-g}}\frac{\partial\sqrt{-g}}{\partial x^0} = \frac{\sqrt{1-kr^2}}{R^3r^2\sin\theta}\frac{R^2\sin\theta}{\sqrt{1-kr^2}}3R^2\dot{R} = 3\frac{\dot{R}}{R}$$

giving

$$(\ln \sqrt{-g})_{,00} = 3\frac{\partial}{\partial t}(\frac{\dot{R}}{R}) = 3\frac{R\ddot{R}-\dot{R}^2}{R^2} = 3\frac{\ddot{R}}{R} - 3(\frac{\dot{R}}{R})^2.$$

We finally have

$$R_{00} = -3\frac{\ddot{R}}{R}. \tag{5.167}$$

One can similaraly show that

$$R_{11} = \frac{R\ddot{R} - 2\dot{R}^2 + 2k}{1-kr^2} \tag{5.168}$$

$$R_{22} = r^2(R\ddot{R} + 2\dot{R}^2 + 2k) \tag{5.169}$$

$$R_{33} = r^2\sin^2\theta(R\ddot{R} + 2\dot{R}^2 + 2k) \tag{5.170}$$

(**do Problem 5.4**)

5.2.3 Riemann Scalar and Einstein Tensor

We now calculate the Ricci scalar $\mathcal{R} \equiv R\mathcal{R}^\alpha_\alpha \equiv g^{\alpha\beta}\mathcal{R}_{\alpha\beta}$. The only non-zero contributions are

$$\begin{aligned}\mathcal{R} &= g^{00}R_{00} + g^{11}R_{11} + g^{22}R_{22} + g^{33}R_{33} &(5.171)\\ &= -6[\frac{\ddot{R}}{R} + (\frac{\dot{R}}{R})^2 + \frac{k}{R^2}] &(5.172)\end{aligned}$$

(**do Problem 5.5**). Finally we calculate the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu}$. The only non-zero component are for $\mu = \nu$. We obtain

$$G_{00} = 3[(\frac{\dot{R}}{R})^2 + \frac{k}{R^2}] \tag{5.173}$$

$$G_{11} = \frac{-1}{1-kr^2}(2\ddot{R}R + \dot{R}^2 + k) \tag{5.174}$$

$$G_{22} = -r^2(2\ddot{R}R + \dot{R}^2 + k) \tag{5.175}$$

$$G_{33} = -r^2\sin^2\theta(2\ddot{R}R + \dot{R}^2 + k) \tag{5.176}$$

(**do Problem 5.6**).

5.2.4 Energy-Momentum Tensor

For a perfect fluid the energy momentum tensor is given in equation (4.26) as

$$T_{\mu\nu} = (\rho + p)u_{\mu\nu} - p\eta_{\mu\nu} \tag{5.177}$$

The tensor for $T^{\mu\nu}$ is written is (4.28) for the metric of Special Relativity. For an arbitrary metric in General Relativity we have

$$T_{\mu\nu} = (\rho + p)u_{\mu\nu} - pg_{\mu\nu} \tag{5.178}$$

where we shall use $g_{\mu\nu}$ from our FRW model. For a *motionless fluid* recall that $u^\mu = (c, \mathbf{0})$ or $U_\mu = (c, -\mathbf{0}) = (c, \mathbf{0}) = (1, \mathbf{0})$ for $c \equiv 1$. Thus

$$T_{00} = \rho + p - p = \rho \tag{5.179}$$

and

$$T_{ii} = -pg_{ii} \tag{5.180}$$

because $u_i = 0$. Upon substitution of the FRW values for the metric given is equations (5.145)-(5.148) we have

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p\frac{R^2}{1-kr^2} & 0 & 0 \\ 0 & 0 & pR^2r^2 & 0 \\ 0 & 0 & 0 & pR^2r^2\sin^2\theta \end{pmatrix} \tag{5.181}$$

5.2.5 Friedmann Equations

Finally we substitute our results into the Einstein field equations $G_{\mu\nu} = 8\pi G T_{\mu\nu} + \Lambda g_{\mu\nu}$.

The $\mu\nu = 00$ component is

$$3[(\frac{\dot{R}}{R})^2 + \frac{k}{R^2}] = 8\pi G\rho + \Lambda$$

giving

$$H^2 \equiv (\frac{\dot{R}}{R})^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} + \frac{\Lambda}{3}. \tag{5.182}$$

The $\mu\nu = 11$ component is

$$\frac{-1}{1-kr^2}(2\ddot{R}R + \dot{R}^2 + k) = 8\pi Gp\frac{R^2}{1-kr^2} + \Lambda\frac{-R^2}{1-kr^2}$$

giving

$$2\frac{\ddot{R}}{R} + (\frac{\dot{R}}{R})^2 + \frac{k}{R^2} = -8\pi Gp + \Lambda.$$

But we now use our previous result (5.182) to give

$$2\frac{\ddot{R}}{R} + \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} = -8\pi Gp + \Lambda$$

to finally give

$$\frac{\ddot{R}}{R} = \frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \tag{5.183}$$

(**do Problem 5.7**).

5.3 Problems

5.1 For the FRW metric show that $\Gamma^1_{33} - (1 - kr^2)r\sin^2\theta$ and $\Gamma = \cot\theta$.

5.2 Show that, for example, $\Gamma^1_{22} = \Gamma^2_{23} = 0$ for the FRW metric.

5.3 Show that $R_{22} = r^2(R\ddot{R} + 2\cot R^2 + 2k)$ for the FRW metric.

5.4 Show that the Ricci scalar is $\mathcal{R} = -6[\frac{\ddot{R}}{R} + (\frac{\dot{R}}{R})^2 + \frac{k}{R^2}]$ for the FRW metric.

5.5 Calculate $G_{\mu\nu}$ for the FRW metric.

5.6 Show that the $\mu\nu = 22$ and $\mu\nu = 33$ components of the Einstein's equations for the FRW metric yield the *same* equation (5.63) as the $\mu\nu = 11$ component.

Chapter 6

Einstein Field Equations

$G^{\mu\nu} = kT^{\mu\nu}$

go through history e.g. he first tried $R^{\mu\nu} = kT^{\mu\nu}$ etc

Chapter 7

Weak Field Limit

derivation of $G^{\mu\nu} = kT^{\mu\nu}$ from equiv princ.

Chapter 8

Lagrangian Methods

Lagrangians for $G^{\mu\nu}$ etc.

(NNNN have assumed special relativity $g_{00} = +1$) (NNN $\bigtriangledown\phi$ term seems to disagree with Kolb and Turner Pg. 276 eqn 8.20).

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