

**CLASSICAL ELECTRODYNAMICS**  
**for Undergraduates**

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1997

# Contents

<b>1</b>	<b>MATRICES</b>	<b>5</b>
1.1	Einstein Summation Convention . . . . .	5
1.2	Coupled Equations and Matrices . . . . .	6
1.3	Determinants and Inverse . . . . .	8
1.4	Solution of Coupled Equations . . . . .	11
1.5	Summary . . . . .	11
1.6	Problems . . . . .	13
1.7	Answers . . . . .	14
1.8	Solutions . . . . .	15
<b>2</b>	<b>VECTORS</b>	<b>19</b>
2.1	Basis Vectors . . . . .	19
2.2	Scalar Product . . . . .	20
2.3	Vector Product . . . . .	22
2.4	Triple and Mixed Products . . . . .	25
2.5	Div, Grad and Curl (differential calculus for vectors) . . . . .	26
2.6	Integrals of Div, Grad and Curl . . . . .	31
	2.6.1 Fundamental Theorem of Gradients . . . . .	32
	2.6.2 Gauss' theorem (Fundamental theorem of Divergence) . . . . .	34
	2.6.3 Stokes' theorem (Fundamental theorem of curl) . . . . .	35
2.7	Potential Theory . . . . .	36
2.8	Curvilinear Coordinates . . . . .	37
	2.8.1 Plane Cartesian (Rectangular) Coordinates . . . . .	37
	2.8.2 Three dimensional Cartesian Coordinates . . . . .	38
	2.8.3 Plane (2-dimensional) Polar Coordinates . . . . .	38
	2.8.4 Spherical (3-dimensional) Polar Coordinates . . . . .	40
	2.8.5 Cylindrical (3-dimensional) Polar Coordinates . . . . .	41
	2.8.6 Div, Grad and Curl in Curvilinear Coordinates . . . . .	43

2.9	Summary . . . . .	43
2.10	Problems . . . . .	44
2.11	Answers . . . . .	46
2.12	Solutions . . . . .	47
2.13	Figure captions for chapter 2 . . . . .	51
<b>3</b>	<b>MAXWELL'S EQUATIONS</b>	<b>53</b>
3.1	Maxwell's equations in differential form . . . . .	54
3.2	Maxwell's equations in integral form . . . . .	56
3.3	Charge Conservation . . . . .	57
3.4	Electromagnetic Waves . . . . .	58
3.5	Scalar and Vector Potential . . . . .	60
<b>4</b>	<b>ELECTROSTATICICS</b>	<b>63</b>
4.1	Equations for electrostatics . . . . .	63
4.2	Electric Field . . . . .	66
4.3	Electric Scalar potential . . . . .	68
4.4	Potential Energy . . . . .	70
4.4.1	Arbitrariness of zero point of potential energy . . . . .	74
4.4.2	Work done in assembling a system of charges . . . . .	74
4.5	Multipole Expansion . . . . .	76
<b>5</b>	<b>Magnetostatics</b>	<b>77</b>
5.1	Equation for Magnetostatics . . . . .	77
5.1.1	Equations from Ampère's Law . . . . .	78
5.1.2	Equations from Gauss' Law . . . . .	78
5.2	Magnetic Field from the Biot-Savart Law . . . . .	79
5.3	Magnetic Field from Ampère's Law . . . . .	81
5.4	Magnetic Field from Vector Potential . . . . .	81
5.5	Units . . . . .	81
<b>6</b>	<b>ELECTRO- AND MAGNETOSTATICS IN MATTER</b>	<b>83</b>
6.1	Units . . . . .	83
6.2	Maxwell's Equations in Matter . . . . .	85
6.2.1	Electrostatics . . . . .	85
6.2.2	Magnetostatics . . . . .	86
6.2.3	Summary of Maxwell's Equations . . . . .	88
6.3	Further Dimensional of Electrostatics . . . . .	89
6.3.1	Dipoles in Electric Field . . . . .	89

<i>CONTENTS</i>	3
6.3.2 Energy Stored in a Dielectric . . . . .	90
6.3.3 Potential of a Polarized Dielectric . . . . .	91
<b>7 ELECTRODYNAMICS AND MAGNETODYNAMICS</b>	<b>93</b>
7.0.4 Faradays's Law of Induction . . . . .	93
7.0.5 Analogy between Faraday field and Magnetostatics . .	96
7.1 Ohm's Law and Electrostatic Force . . . . .	97
<b>8 MAGNETOSTATICS</b>	<b>101</b>
<b>9 ELECTRO- &amp; MAGNETOSTATICS IN MATTER</b>	<b>103</b>
<b>10 ELECTRODYNAMICS AND MAGNETODYNAMICS</b>	<b>105</b>
<b>11 ELECTROMAGNETIC WAVES</b>	<b>107</b>
<b>12 SPECIAL RELATIVITY</b>	<b>109</b>



# Chapter 1

## MATRICES

### 1.1 Einstein Summation Convention

Even though we shall not study vectors until chapter 2, we will introduce simple vectors now so that we can more easily understand the Einstein summation convention.

We are often used to writing vectors in terms of unit basis vectors as

$$\mathbf{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}. \quad (1.1)$$

(see Figs. 2.7 and 2.8.) However we will find it much more convenient instead to write this as

$$\mathbf{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 \quad (1.2)$$

where our components  $(A_x, A_y, A_z)$  are re-written as  $(A_1, A_2, A_3)$  and the basis vectors  $(\hat{i}, \hat{j}, \hat{k})$  become  $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ . This is more natural when considering other dimensions. For instance in 2 dimensions we would write  $\mathbf{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2$  and in 5 dimensions we would write  $\mathbf{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 + A_4 \hat{e}_4 + A_5 \hat{e}_5$ .

However, even this gets a little clumsy. For example in 10 dimensions we would have to write out 10 terms. It is much easier to write

$$\mathbf{A} = \sum_i^N A_i \hat{e}_i \quad (1.3)$$

where  $N$  is the number of dimensions. Notice in this formula that the index  $i$  occurs twice in the expression  $A_i \hat{e}_i$ . Einstein noticed this always occurred and so whenever an index was repeated twice he simply didn't bother to

write  $\sum_i^N$  as well because he just knew it was always there for twice repeated indices so that instead of writing  $\mathbf{A} = \sum_i A_i \hat{e}_i$  he would simply write  $\mathbf{A} = A_i \hat{e}_i$  knowing that there was really a  $\sum_i$  in the formula, that he wasn't bothering to write explicitly. Thus the Einstein summation convention is defined generally as

$$X_i Y_i \equiv \sum_i^N X_i Y_i \quad (1.4)$$

Let us work out some examples.

---

**Example 1.1.1** What is  $A_i B_i$  in 2 dimensions ?

**Solution**  $A_i B_i \equiv \sum_{i=1}^2 A_i B_i = A_1 B_1 + A_2 B_2$

**Example 1.1.2** What is  $A_{ij} B_{jk}$  in 3 dimensions ?

**Solution** We have 3 indices here  $(i, j, k)$ , but only  $j$  is repeated twice and so  $A_{ij} B_{jk} \equiv \sum_{j=1}^3 A_{ij} B_{jk} = A_{i1} B_{1k} + A_{i2} B_{2k} + A_{i3} B_{3k}$

---

## 1.2 Coupled Equations and Matrices

Consider the two simultaneous (or coupled) equations

$$\begin{aligned} x + y &= 2 \\ x - y &= 0 \end{aligned} \quad (1.5)$$

which have the solutions  $x = 1$  and  $y = 1$ . A *different* way of writing these coupled equations is in terms of objects called *matrices*,

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + y \\ x - y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad (1.6)$$

Notice how the two matrices on the far left hand side get multiplied together. The multiplication rule is perhaps clearer if we write

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} \quad (1.7)$$

We have invented these matrices with their rule of 'multiplication' simply as a way of writing (1.5) in a fancy form. If we had 3 simultaneous equations

$$\begin{aligned}x + y + z &= 3 \\x - y + z &= 1 \\2x + z &= 3\end{aligned}\tag{1.8}$$

we would write

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ x - y + z \\ 2x + 0y + z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix}\tag{1.9}$$

Thus matrix notation is simply a way of writing down simultaneous equations. In the far left hand side of (1.6), (1.7) and (1.9) we have a square matrix multiplying a column matrix. Equation (1.6) could also be written as

$$[A][X] = [B]\tag{1.10}$$

with

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \equiv \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix} \equiv \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}\tag{1.11}$$

or  $B_1 = A_{11}x_1 + A_{12}x_2$  and  $B_2 = A_{21}x_1 + A_{22}x_2$ . A shorthand for this is

$$\boxed{B_i = A_{ik}x_k}\tag{1.12}$$

which is just a shorthand way of writing matrix multiplication form. Note  $x_k$  has 1 index and is a vector. Thus vectors are often written  $\mathbf{x} = x\hat{i} + y\hat{j}$  or just  $\begin{pmatrix} x \\ y \end{pmatrix}$ . This is the matrix way of writing a vector. (**do Problem 1.1**)

Sometimes we want to multiply two square matrices together. The rule for doing this is

$$\begin{aligned}& \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \\ \equiv & \begin{pmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{pmatrix} \equiv \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\end{aligned}\tag{1.13}$$



Thus, for example,  $C_{11} = A_{11}B_{11} + A_{12}B_{21}$  and  $C_{21} = A_{21}B_{11} + A_{22}B_{21}$  which can be written in shorthand as

$$\boxed{C_{ij} = A_{ik}B_{kj}} \tag{1.14}$$

which is the matrix multiplication formula for square matrices. This is very easy to understand as it is just a generalization of (1.12) with an extra index  $j$  tacked on. (**do Problems 1.2 and 1.3**)

**Example 1.2.1** Show that equation (1.14) gives the correct form for  $C_{21}$ .

**Solution**  $C_{ij} = A_{ik}B_{kj}$ . Thus  $C_{21} = A_{2k}B_{k1} = A_{21}B_{11} + A_{22}B_{21}$ .

**Example 1.2.2** Show that  $C_{ij} = A_{ik}B_{jk}$  is the *wrong* formula for matrix multiplication.

**Solution** Let's work it out for  $C_{21}$ :

$C_{21} = A_{2k}B_{1k} = A_{21}B_{11} + A_{22}B_{12}$ . Comparing to the expressions above we can see that the second term is wrong here.

### 1.3 Determinants and Inverse

We now need to discuss matrix determinant and matrix inverse. The determinant for a  $2 \times 2$  matrix is denoted

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} \equiv A_{11}A_{22} - A_{21}A_{12} \tag{1.15}$$

and for a  $3 \times 3$  matrix it is

$$\begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix} \equiv A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} \\ - A_{31}A_{22}A_{13} - A_{21}A_{12}A_{33} - A_{11}A_{32}A_{23} \tag{1.16}$$

The identity matrix  $[I]$  is  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$  for  $2 \times 2$  matrices or  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  for  $3 \times 3$  matrices etc.  $I$  is *defined* to have the true property of an identity, namely

$$IB \equiv BI \equiv B \quad (1.17)$$

where  $B$  is any matrix. *Exercise:* Check this is true by multiplying *any*  $2 \times 2$  matrix times  $I$ .

The inverse of a matrix  $A$  is denoted as  $A^{-1}$  and *defined* such that

$$AA^{-1} = A^{-1}A = I. \quad (1.18)$$

The inverse is actually calculated using objects called *cofactors* [3]. Consider the matrix  $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$ . The cofactor of the matrix element  $A_{21}$  for example is defined as

$$\text{cof}(A_{21}) \equiv (-)^{2+1} \begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} = -(A_{12}A_{33} - A_{32}A_{13}) \quad (1.19)$$

The way to get the matrix elements appearing in this determinant is just by crossing out the rows and columns in which  $A_{21}$  appears in matrix  $A$  and the elements left over go into the cofactor.

**Example 1.3.1** What is the cofactor of  $A_{22}$  ?

**Solution**

$$\text{cof}(A_{22}) \equiv (-)^{2+2} \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = A_{11}A_{33} - A_{31}A_{13}$$

Finally we get to the matrix inverse. The matrix elements of the inverse matrix are given by [3]

$$\boxed{(A^{-1})_{ij} \equiv \frac{1}{|A|} \text{cof}(A_{ji})}$$

(1.20)

Notice that the  $ij$  matrix element of the inverse is given by the  $ji$  cofactor.

---

**Example 1.3.2** Find the inverse of  $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$  and check your answer.

**Solution** Let's write  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ .

Now  $\text{cof}(A_{11}) \equiv (-)^{1+1}|A_{22}| = A_{22} = -1$ . Notice that the determinant in the cofactor is just a single number for  $2 \times 2$  matrices. The other cofactors are

$$\text{cof}(A_{12}) \equiv (-)^{1+2}|A_{21}| = -A_{21} = -1$$

$$\text{cof}(A_{21}) \equiv (-)^{2+1}|A_{12}| = -A_{12} = -1$$

$$\text{cof}(A_{22}) \equiv (-)^{2+2}|A_{11}| = A_{11} = 1.$$

The determinant of  $A$  is  $|A| = A_{11}A_{22} - A_{21}A_{12} = -1 - 1 = -2$ .

Thus from (1.20)

$$(A^{-1})_{11} \equiv \frac{1}{-2}\text{cof}(A_{11}) = \frac{1}{2},$$

$$(A^{-1})_{12} \equiv \frac{1}{-2}\text{cof}(A_{21}) = \frac{1}{2},$$

$$(A^{-1})_{21} \equiv \frac{1}{-2}\text{cof}(A_{12}) = \frac{1}{2},$$

$$(A^{-1})_{22} \equiv \frac{1}{-2}\text{cof}(A_{22}) = -\frac{1}{2}.$$

$$\text{Thus } A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

We can check our answer by making sure that  $AA^{-1} = I$  as follows,

$$\begin{aligned} AA^{-1} &= \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus we are now sure that our calculation of  $A^{-1}$  is correct. Also having verified this gives justification for believing in all the formulas for cofactors and inverse given above. **(do Problems 1.4 and 1.5)**

---

## 1.4 Solution of Coupled Equations

By now we have developed quite a few skills with matrices, but what is the point of it all? Well it allows us to solve simultaneous (coupled) equations (such as (1.5)) with matrix methods. Writing (1.6) as  $AX = Y$  where  $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ,  $X = \begin{pmatrix} x \\ y \end{pmatrix}$ ,  $Y = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ , we see that the *solution* we want is the value for  $x$  and  $y$ . In other words we want to find the column matrix  $X$ . We have

$$AX = Y \quad (1.21)$$

so that

$$A^{-1}AX = A^{-1}Y. \quad (1.22)$$

Thus

$$X = A^{-1}Y \quad (1.23)$$

where we have used (1.18).

**Example 1.4.1** Solve the set of coupled equations (1.5) with matrix methods.

**Solution** Equation (1.5) is re-written in (1.6) as  $AX = Y$  with  $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ ,  $X = \begin{pmatrix} x \\ y \end{pmatrix}$ ,  $Y = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ . We want  $X = A^{-1}Y$ . From Example 1.3.2 we have  $A^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ . Thus

$X = A^{-1}Y$  means

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus  $x = 1$  and  $y = 1$ .

## 1.5 Summary

This concludes our discussion of matrices. Just to summarize a little, firstly it is more than reasonable to simply think of matrices as another way of writing and solving simultaneous equations. Secondly, our invention of how to multiply matrices in (1.12) was invented *only* so that it reproduces the

coupled equations (1.5). For multiplying two square matrices equation (1.14) is just (1.12) with another index tacked on. Thirdly, even though we did not *prove* mathematically that the inverse is given by (1.20), nevertheless we can *believe* in the formula because we always found that using it gives  $AA^{-1} = I$ . It doesn't matter *how* you get  $A^{-1}$ , as long as  $AA^{-1} = I$  you know that you have found the right answer. A proof of (1.20) can be found in mathematics books [3, 4].

## 1.6 Problems

1.1 Show that  $B_i = A_{ik}x_k$  gives (1.11).

1.2 Show that  $C_{ij} = A_{ik}B_{kj}$  gives (1.13).

1.3 Show that matrix multiplication is non-commutative, i.e.  $AB \neq BA$ .

1.4 Find the inverse of  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  and check your answer.

1.5 Find the inverse of  $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$  and check your answer.

1.6 Solve the following simultaneous equations with matrix methods:

$$x + y + z = 4$$

$$x - y + z = 2$$

$$2x + z = 4$$

**1.7** Answers

**1.4** 
$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

**1.5** 
$$\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

**1.6**  $x = 1, y = 1, z = 2.$

## 1.8 Solutions

### 1.1

$B_i = A_{ik}x_k$ . We simply evaluate each term. Thus

$$B_1 = A_{1k}x_k = A_{11}x_1 + A_{12}x_2$$

$$B_2 = A_{2k}x_k = A_{21}x_1 + A_{22}x_2.$$

### 1.2

$C_{ij} = A_{ik}B_{kj}$ . Again just evaluate each term. Thus

$$C_{11} = A_{1k}B_{k1} = A_{11}B_{11} + A_{12}B_{21}$$

$$C_{12} = A_{1k}B_{k2} = A_{11}B_{12} + A_{12}B_{22}$$

$$C_{21} = A_{2k}B_{k1} = A_{21}B_{11} + A_{22}B_{21}$$

$$C_{22} = A_{2k}B_{k2} = A_{21}B_{12} + A_{22}B_{22} .$$

### 1.3

This can be seen by just multiplying any two matrices, say

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 19 & 22 \\ 33 & 50 \end{pmatrix} .$$

$$\text{Whereas } \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 23 & 34 \\ 31 & 46 \end{pmatrix} .$$

showing that matrix multiplication does not commute.



## 1.4

Let's write  $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . Now  $\text{cof}(A_{11}) \equiv (-)^{1+1}|A_{22}| = A_{22} = 1$ . Notice that the determinant in the cofactor is just a single number for  $2 \times 2$  matrices. The other cofactors are

$$\text{cof}(A_{12}) \equiv (-)^{1+2}|A_{21}| = -A_{21} = 0$$

$$\text{cof}(A_{21}) \equiv (-)^{2+1}|A_{12}| = -A_{12} = -1$$

$$\text{cof}(A_{22}) \equiv (-)^{2+2}|A_{11}| = A_{11} = 1.$$

The determinant of  $A$  is  $|A| = A_{11}A_{22} - A_{21}A_{12} = 1$ . Thus from (1.20)

$$(A^{-1})_{11} \equiv \frac{1}{1}\text{cof}(A_{11}) = 1,$$

$$(A^{-1})_{12} \equiv \frac{1}{1}\text{cof}(A_{21}) = -1,$$

$$(A^{-1})_{21} \equiv \frac{1}{1}\text{cof}(A_{12}) = 0,$$

$$(A^{-1})_{22} \equiv \frac{1}{1}\text{cof}(A_{22}) = 1.$$

$$\text{Thus } A^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

We can check our answer by making sure that  $AA^{-1} = I$  as follows,

$$AA^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I. \text{ Thus we are now sure that our answer for } A^{-1} \text{ is correct.}$$

**1.5**

Let's write  $A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix}$ .

The cofactors are

$$\text{cof}(A_{11}) = (-)^{1+1} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} = +(A_{22}A_{33} - A_{32}A_{23}) = -1 - 0 = -1$$

$$\text{cof}(A_{12}) = (-)^{1+2} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} = -(A_{21}A_{33} - A_{31}A_{23}) = -1 + 2 = 1$$

$$\text{cof}(A_{13}) = (-)^{1+3} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix} = +(A_{21}A_{32} - A_{31}A_{22}) = 0 + 2 = 2$$

$$\text{cof}(A_{21}) = (-)^{2+1} \begin{vmatrix} A_{12} & A_{13} \\ A_{32} & A_{33} \end{vmatrix} = -(A_{12}A_{33} - A_{32}A_{13}) = -1 + 0 = -1$$

$$\text{cof}(A_{22}) = (-)^{2+2} \begin{vmatrix} A_{11} & A_{13} \\ A_{31} & A_{33} \end{vmatrix} = +(A_{11}A_{33} - A_{31}A_{13}) = 1 - 2 = -1$$

$$\text{cof}(A_{23}) = (-)^{2+3} \begin{vmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{vmatrix} = -(A_{11}A_{32} - A_{31}A_{12}) = -0 + 2 = 2$$

$$\text{cof}(A_{31}) = (-)^{3+1} \begin{vmatrix} A_{12} & A_{13} \\ A_{22} & A_{23} \end{vmatrix} = +(A_{12}A_{23} - A_{22}A_{13}) = 1 + 1 = 2$$

$$\text{cof}(A_{32}) = (-)^{3+2} \begin{vmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{vmatrix} = -(A_{11}A_{23} - A_{21}A_{13}) = -1 + 1 = 0$$

$$\text{cof}(A_{33}) = (-)^{3+3} \begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = +(A_{11}A_{22} - A_{21}A_{12}) = -1 - 1 = -2$$

The determinant of  $A$  is (see equation (1.16))  $|A| = 2$ . Thus from (1.20)

$$(A^{-1})_{11} \equiv \frac{1}{2}\text{cof}(A_{11}) = -\frac{1}{2},$$

$$(A^{-1})_{12} \equiv \frac{1}{2}\text{cof}(A_{21}) = -\frac{1}{2},$$

$$(A^{-1})_{13} \equiv \frac{1}{2}\text{cof}(A_{31}) = 1,$$

$$\begin{aligned} (A^{-1})_{21} &\equiv \frac{1}{2} \text{cof}(A_{12}) = \frac{1}{2}, \\ (A^{-1})_{22} &\equiv \frac{1}{2} \text{cof}(A_{22}) = -\frac{1}{2}, \\ (A^{-1})_{23} &\equiv \frac{1}{2} \text{cof}(A_{32}) = 0, \\ (A^{-1})_{31} &\equiv \frac{1}{2} \text{cof}(A_{13}) = 1, \\ (A^{-1})_{32} &\equiv \frac{1}{2} \text{cof}(A_{23}) = 1, \\ (A^{-1})_{33} &\equiv \frac{1}{2} \text{cof}(A_{33}) = -1, \end{aligned}$$

$$\text{Thus } A^{-1} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

We can check our answer by making sure that  $AA^{-1} = I$  as follows,

$$AA^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I. \text{ Thus}$$

we are now sure that our answer for  $A^{-1}$  is correct.

### 1.6

$AX = Y$  is written as

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix}$$

Thus  $X = A^{-1}Y$  and we found  $A^{-1}$  in problem 1.5.

$$\text{Thus } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

Thus the solution is  $x = 1, y = 1, z = 2$ .

## Chapter 2

# VECTORS

In this review chapter we shall assume some familiarity with vectors from freshman physics. Extra details can be found in the references [1].

### 2.1 Basis Vectors

We shall assume that we can always write a vector in terms of a set of basis vectors

$$\mathbf{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 = A_i \hat{e}_i \quad (2.1)$$

where the components of the vector  $\mathbf{A}$  are  $(A_x, A_y, A_z)$  or  $(A_1, A_2, A_3)$  and the basis vectors are  $(\hat{i}, \hat{j}, \hat{k})$  or  $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ . The index notation  $A_i$  and  $\hat{e}_i$  is preferred because it is easy to handle any number of dimensions.

In equation (2.1) we are using the Einstein summation convention for repeated indices which says that

$$x_i y_i \equiv \sum_i x_i y_i. \quad (2.2)$$

In other words when indices are repeated it means that a sum is always implied. We shall use this convention throughout this book.

With basis vectors it is always easy to add and subtract vectors

$$\mathbf{A} + \mathbf{B} = A_i \hat{e}_i + B_i \hat{e}_i = (A_i + B_i) \hat{e}_i. \quad (2.3)$$

Thus the components of  $\mathbf{A} + \mathbf{B}$  are obtained by adding the components of  $\mathbf{A}$  and  $\mathbf{B}$  separately. Similarly, for example,

$$\mathbf{A} - 2\mathbf{B} = A_i \hat{e}_i - 2B_i \hat{e}_i = (A_i - 2B_i) \hat{e}_i. \quad (2.4)$$

## 2.2 Scalar Product

Let us define the scalar product (also often called the inner product) of two vectors as

$$\boxed{\mathbf{A} \cdot \mathbf{B} \equiv AB \cos \theta} \tag{2.5}$$

where  $A \equiv |\mathbf{A}|$  is the magnitude of  $\mathbf{A}$  and  $\theta$  is the angle between  $\mathbf{A}$  and  $\mathbf{B}$ . (Here  $||$  means magnitude and not determinant). Thus

$$\mathbf{A} \cdot \mathbf{B} = A_i \hat{e}_i \cdot B_j \hat{e}_j \tag{2.6}$$

Note that when we are multiplying two quantities that both use the Einstein summation convention, we must use *different* indices. (We were OK before when adding). To see this let's work out (2.6) explicitly in two dimensions.  $\mathbf{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2$  and  $\mathbf{B} = B_1 \hat{e}_1 + B_2 \hat{e}_2$  so that  $\mathbf{A} \cdot \mathbf{B} = (A_1 \hat{e}_1 + A_2 \hat{e}_2) \cdot (B_1 \hat{e}_1 + B_2 \hat{e}_2) = A_1 B_1 \hat{e}_1 \cdot \hat{e}_1 + A_1 B_2 \hat{e}_1 \cdot \hat{e}_2 + A_2 B_1 \hat{e}_2 \cdot \hat{e}_1 + A_2 B_2 \hat{e}_2 \cdot \hat{e}_2$  which is exactly what you get when expanding out (2.6). However if we had mistakenly written  $\mathbf{A} \cdot \mathbf{B} = A_i \hat{e}_i \cdot B_i \hat{e}_i = A_i B_i \hat{e}_i \cdot \hat{e}_i = A_1 B_1 \hat{e}_1 \cdot \hat{e}_1 + A_2 B_2 \hat{e}_2 \cdot \hat{e}_2$  which is wrong. A basic rule of thumb is that it's OK to have double repeated indices but it's never OK to have more.

Let's return to our scalar product

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= A_i \hat{e}_i \cdot B_j \hat{e}_j \\ &= (\hat{e}_i \cdot \hat{e}_j) A_i B_j \\ &\equiv g_{ij} A_i B_j \end{aligned} \tag{2.7}$$

where we define a quantity  $g_{ij}$  called the metric tensor as  $g_{ij} \equiv \hat{e}_i \cdot \hat{e}_j$ . Note that vector components  $A_i$  have one index, scalars never have any indices and matrix elements have two indices  $A_{ij}$ . Thus scalars are called tensors of rank zero, vectors are called tensors of rank one and *some* matrices are tensors of rank two. Not all matrices are tensors because they must also satisfy the tensor transformation rules [1] which we will not go into here. However all tensors of rank two can be written as matrices. There are also tensors of rank three  $A_{ijk}$  etc. Tensors of rank three are called tensors of rank three. They do not have a special name like scalar and vector. The same is true for tensors of rank higher than three.

Now *if* we choose our basis vectors  $\hat{e}_i$  to be of unit length  $|\hat{e}_i| = 1$  and orthogonal to each other then by (2.5)

$$\hat{e}_i \cdot \hat{e}_j = |\hat{e}_i| |\hat{e}_j| \cos \theta = \cos \theta = \delta_{ij} \quad (2.8)$$

where  $\delta_{ij}$  is defined as

$$\begin{aligned} \delta_{ij} &\equiv 1 & \text{if} & \quad i = j \\ &\equiv 0 & \text{if} & \quad i \neq j \end{aligned} \quad (2.9)$$

which can also be written as a matrix

$$[\delta_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (2.10)$$

for two dimensions. Thus if  $g_{ij} = \delta_{ij}$  then we have what is called a Cartesian space (or flat space), so that

$$\begin{aligned} \mathbf{A} \cdot \mathbf{B} &= g_{ij} A_i B_j = \delta_{ij} A_i B_j \\ &= \delta_{i1} A_i B_1 + \delta_{i2} A_i B_2 \\ &= \delta_{11} A_1 B_1 + \delta_{21} A_2 B_1 + \delta_{12} A_1 B_2 + \delta_{22} A_2 B_2 \\ &= A_1 B_1 + A_2 B_2 \equiv A_i B_i. \end{aligned} \quad (2.11)$$

Thus

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A_i B_i} \quad (2.12)$$

Now  $\mathbf{A} \cdot \mathbf{B} = A_i B_i = A_x B_x + A_y B_y$  is just the scalar product that we are used to from freshman physics, and so Pythagoras' theorem follows as

$$\mathbf{A} \cdot \mathbf{A} \equiv A^2 = A_i A_i = A_x^2 + A_y^2. \quad (2.13)$$

Note that the fundamental relations of the scalar product (2.12) and the form of Pythagoras' theorem follow directly from our specification of the

metric tensor as  $g_{ij} = \delta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

As an aside, note that we could easily have defined a non-Cartesian space, for example  $g_{ij} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  in which case Pythagoras' theorem would change to

$$\mathbf{A} \cdot \mathbf{A} \equiv A^2 = A_i A_i = A_x^2 + A_y^2 + A_x A_y. \quad (2.14)$$

Thus it is the metric tensor  $g_{ij} \equiv \hat{e}_i \cdot \hat{e}_j$  given by the scalar product of the unit vectors which (almost) completely defines the vector space that we are considering.

## 2.3 Vector Product

In the previous section we 'multiplied' two vectors to get a scalar  $\mathbf{A} \cdot \mathbf{B}$ . However if we start with two vectors maybe we can also define a 'multiplication' that results in a vector, which is our only other choice. This is called the vector product or cross product denoted as  $\mathbf{A} \times \mathbf{B}$ . The magnitude of the vector product is *defined* as

$$|\mathbf{A} \times \mathbf{B}| \equiv AB \sin \theta \quad (2.15)$$

whereas the direction of  $\mathbf{C} = \mathbf{A} \times \mathbf{B}$  is *defined* as being given by the *right hand rule*, whereby you hold the thumb, fore-finger and middle finger of your right hand all at right angles to each other. The thumb represents vector  $\mathbf{C}$ , the fore-finger represents  $\mathbf{A}$  and the middle finger represents  $\mathbf{B}$ .

**Example 2.3.1** If  $\mathbf{D}$  is a vector pointing to the right of the page and  $\mathbf{E}$  points down the page, what is the direction of  $\mathbf{D} \times \mathbf{E}$  and  $\mathbf{E} \times \mathbf{D}$ ?

**Solution**  $\mathbf{D}$  is the fore-finger,  $\mathbf{E}$  is the middle finger and so  $\mathbf{D} \times \mathbf{E}$  which is represented by the thumb ends up pointing into the page. For  $\mathbf{E} \times \mathbf{D}$  we swap fingers and the thumb  $\mathbf{E} \times \mathbf{D}$  points out of the page. (**do Problem 2.1**)

From our definitions above for the magnitude and direction of the vector product it follows that

$$\hat{e}_1 \times \hat{e}_1 = \hat{e}_2 \times \hat{e}_2 = \hat{e}_3 \times \hat{e}_3 = 0 \quad (2.16)$$

because  $\theta = 0^\circ$  for these cases. Also

$$\begin{aligned} \hat{e}_1 \times \hat{e}_2 &= \hat{e}_3 = -\hat{e}_2 \times \hat{e}_1 \\ \hat{e}_2 \times \hat{e}_3 &= \hat{e}_1 = -\hat{e}_3 \times \hat{e}_2 \\ \hat{e}_3 \times \hat{e}_1 &= \hat{e}_2 = -\hat{e}_1 \times \hat{e}_3 \end{aligned} \quad (2.17)$$

which follows from the right hand rule and also because  $\theta = 90^\circ$ .

Let us now introduce some short hand notation in the form of the Levi-Civita symbol (not a tensor ??) defined as

$$\begin{aligned}\epsilon_{ijk} &\equiv +1 && \text{if } ijk \text{ are in the order of } 123 \text{ (even permutation)} \\ &\equiv -1 && \text{if } ijk \text{ not in the order of } 123 \text{ (odd permutation)} \\ &\equiv 0 && \text{if any of } ijk \text{ are repeated (not a permutation)}\end{aligned}\quad (2.18)$$

For example  $\epsilon_{123} = +1$  because 123 are in order. Also  $\epsilon_{231} = +1$  because the numbers are still in order whereas  $\epsilon_{312} = -1$  because 312 are out of numerical sequence.  $\epsilon_{122} = \epsilon_{233} = 0$  etc., because the numbers are not a permutation of 123 because two numbers are repeated. Also note that  $\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{ikj}$  etc. That is we can permute the indices without changing the answer, but we get a minus sign if we only swap the places of two indices. **(do Problem 2.2)**

We can now write down the vector product of our basis vectors as

$$\hat{e}_i \times \hat{e}_j = \epsilon_{ijk} \hat{e}_k \quad (2.19)$$

where a sum over  $k$  is implied because it is a repeated index.

**Example 2.3.2** Using (2.19) show that  $\hat{e}_2 \times \hat{e}_3 = \hat{e}_1$  and  $\hat{e}_2 \times \hat{e}_1 = -\hat{e}_3$  and  $\hat{e}_1 \times \hat{e}_1 = 0$ .

**Solution** From (2.19) we have

$$\begin{aligned}\hat{e}_2 \times \hat{e}_3 &= \epsilon_{23k} \hat{e}_k \\ &= \epsilon_{231} \hat{e}_1 + \epsilon_{232} \hat{e}_2 + \epsilon_{233} \hat{e}_3 \\ &= +1 \hat{e}_1 + 0 \hat{e}_2 + 0 \hat{e}_3 \\ &= \hat{e}_1.\end{aligned}$$

$$\begin{aligned}\hat{e}_2 \times \hat{e}_1 &= \epsilon_{21k} \hat{e}_k \\ &= \epsilon_{211} \hat{e}_1 + \epsilon_{212} \hat{e}_2 + \epsilon_{213} \hat{e}_3 \\ &= 0 \hat{e}_1 + 0 \hat{e}_2 - 1 \hat{e}_3 \\ &= -\hat{e}_3.\end{aligned}$$

$$\begin{aligned}\hat{e}_1 \times \hat{e}_1 &= \epsilon_{11k} \hat{e}_k \\ &= 0.\end{aligned}$$

because  $\epsilon_{11k} = 0$  no matter what the value of  $k$ .



Using (2.19) we can now write the vector product as  $\mathbf{A} \times \mathbf{B} = A_i \hat{e}_i \times B_j \hat{e}_j = A_i B_j \hat{e}_i \times \hat{e}_j = A_i B_j \epsilon_{ijk} \hat{e}_k$ . Thus our general formula for the vector product is

$$\boxed{\mathbf{A} \times \mathbf{B} = \epsilon_{ijk} A_i B_j \hat{e}_k} \tag{2.20}$$

The advantage of this formula is that it gives *both* the magnitude and direction. Note that the  $k$ th component of  $\mathbf{A} \times \mathbf{B}$  is just the coefficient in front of  $\hat{e}_k$ , i.e.

$$(\mathbf{A} \times \mathbf{B})_k = \epsilon_{ijk} A_i B_j \tag{2.21}$$

**Example 2.3.3** Evaluate the  $x$  component of (2.20) explicitly.

**Solution** The right hand side of (2.20) has 3 sets of twice repeated indices  $ijk$  which implies  $\sum_i \sum_j \sum_k$ . Let's do  $\sum_k$  first.

$\mathbf{A} \times \mathbf{B} = \epsilon_{ij1} A_i B_j \hat{e}_1 + \epsilon_{ij2} A_i B_j \hat{e}_2 + \epsilon_{ij3} A_i B_j \hat{e}_3$ . The  $x$  component of  $\mathbf{A} \times \mathbf{B}$  is just the coefficient in front of  $\hat{e}_1$ . Let's do the sum over  $i$  first. Thus

$$\begin{aligned} (\mathbf{A} \times \mathbf{B})_1 &= \epsilon_{ij1} A_i B_j \\ &= \epsilon_{1j1} A_1 B_j + \epsilon_{2j1} A_2 B_j + \epsilon_{3j1} A_3 B_j + \\ &= \epsilon_{111} A_1 B_1 + \epsilon_{121} A_1 B_2 + \epsilon_{131} A_1 B_3 + \\ &\quad \epsilon_{211} A_2 B_1 + \epsilon_{221} A_2 B_2 + \epsilon_{231} A_2 B_3 + \\ &\quad \epsilon_{311} A_3 B_1 + \epsilon_{321} A_3 B_2 + \epsilon_{331} A_3 B_3 \\ &= 0A_1 B_1 + 0A_1 B_2 + 0A_1 B_3 + \\ &\quad 0A_2 B_1 + 0A_2 B_2 + 1A_2 B_3 + \\ &\quad 0A_3 B_1 - 1A_3 B_2 + 0A_3 B_3 \\ &= A_2 B_3 - A_3 B_2. \end{aligned}$$

Similarly one can show (**do Problem 2.3**) that  $(\mathbf{A} \times \mathbf{B})_2 = A_3B_1 - A_1B_3$  and  $(\mathbf{A} \times \mathbf{B})_3 = A_1B_2 - A_2B_1$ , or in other words

$$\mathbf{A} \times \mathbf{B} = (A_2B_3 - A_3B_2)\hat{e}_1 + (A_3B_1 - A_1B_3)\hat{e}_2 + (A_1B_2 - A_2B_1)\hat{e}_3 \quad (2.22)$$

which is simply the result of working out (2.20). The formula (2.20) can perhaps be more easily memorized by writing it as a determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \quad (2.23)$$

This doesn't come from the theory of matrices or anything complicated. It is just a memory device for getting (2.20). (**do Problem 2.4**)

## 2.4 Triple and Mixed Products

We will now consider triples and mixtures of scalar and vector products such as  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$  and  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$  etc. Our Kronecker delta  $\delta_{ij}$  and Levi-Civita  $\epsilon_{ijk}$  symbols will make these *much* easier to work out. We shall simply show a few examples and you can do some problems. However before proceeding there is a useful identity that we shall use namely

$$\epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \quad (2.24)$$

To show this is true it's easiest to just work out the left and right hand sides for special index values.

**Example 2.4.1** Show that (2.24) is true for  $i = 1, j = 2, l = 3, m = 3$ .

**Solution** The left hand side is

$$\begin{aligned} \epsilon_{kij}\epsilon_{klm} &= \epsilon_{k12}\epsilon_{k33} \\ &= \epsilon_{112}\epsilon_{133} + \epsilon_{212}\epsilon_{233} + \epsilon_{312}\epsilon_{333} \\ &= (0)(0) + (0)(0) + (+1)(0) \\ &= 0. \end{aligned} \quad (2.25)$$

The right hand side is

$$\begin{aligned} \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} &= \delta_{13}\delta_{23} - \delta_{13}\delta_{23} \\ &= (0)(0) - (0)(0) \\ &= 0. \end{aligned} \quad (2.26)$$

Thus the left hand side equals the right hand side. (**do Problem 2.5**)

**Example 2.4.2** Show that  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$ .

**Solution**

$$\begin{aligned}
 \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= A_k (\mathbf{B} \times \mathbf{C})_k = A_k \epsilon_{ijk} B_i C_j \\
 &= B_i \epsilon_{ijk} C_j A_k \\
 &= B_i \epsilon_{jki} C_j A_k \\
 &= B_i (\mathbf{C} \times \mathbf{A})_i \\
 &= \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})
 \end{aligned}
 \tag{2.27}$$

where we used the fact that  $\epsilon_{ijk} = \epsilon_{jki}$ . (**do Problem 2.6**)

## 2.5 Div, Grad and Curl (differential calculus for vectors)

Some references for this section are the books by Griffiths and Arfken [2, 8]. We have learned about the derivative  $\frac{df(x)}{dx}$  of the function  $f(x)$  in elementary calculus. Recall that the derivative gives us the rate of change of the function in the direction of  $x$ . If  $f$  is a function of three variables  $f(x, y, z)$  we can form partial derivatives  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$ ,  $\frac{\partial f}{\partial z}$  which tell us the rate of change of  $f$  in three different directions. Recall that the vector components  $A_x$ ,  $A_y$ ,  $A_z$  tell us the size of the vector  $\mathbf{A}$  in three different directions. Thus we expect that the three partial derivatives may be interpreted as the components of a *vector* derivative denoted by  $\nabla$  and defined by

$$\boxed{\nabla \equiv \hat{e}_i \frac{\partial}{\partial x_i} \equiv \hat{e}_i \nabla_i}
 \tag{2.28}$$

with  $\nabla_i \equiv \frac{\partial}{\partial x_i}$ . Expanding (with the Einstein summation convention) gives  $\nabla = \hat{e}_1 \frac{\partial}{\partial x_1} + \hat{e}_2 \frac{\partial}{\partial x_2} + \hat{e}_3 \frac{\partial}{\partial x_3} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$ . (Note that it's important to write the  $\frac{\partial}{\partial x_i}$  to the right of the  $\hat{e}_i$ , otherwise we might mistakenly think

## 2.5. DIV, GRAD AND CURL (DIFFERENTIAL CALCULUS FOR VECTORS) 27

that  $\frac{\partial}{\partial x_i}$  acts on  $\hat{e}_i$ .) Even though we use the standard vector symbol for  $\nabla$  it is not quite the same type of vector as  $\mathbf{A}$ . Rather  $\nabla$  is a vector derivative or vector operator. Actually its proper mathematical name is a dual vector or a one-form.

Now that we are armed with this new vector operator weapon, let's start using it. The two types of objects that we know about so far are functions  $\phi(x, y, z)$  and vectors  $\mathbf{A}$ . The only object we can form with  $\nabla$  acting on  $\phi$  is called the *gradient* of the function  $\phi$  and is defined by

$$\nabla\phi \equiv \hat{e}_i \frac{\partial\phi}{\partial x_i} \quad (2.29)$$

which is expanded as  $\nabla\phi \equiv \hat{e}_i \frac{\partial\phi}{\partial x_i} = \hat{e}_1 \frac{\partial\phi}{\partial x_1} + \hat{e}_2 \frac{\partial\phi}{\partial x_2} + \hat{e}_3 \frac{\partial\phi}{\partial x_3} = \hat{i} \frac{\partial\phi}{\partial x} + \hat{j} \frac{\partial\phi}{\partial y} + \hat{k} \frac{\partial\phi}{\partial z}$ .

Now let's consider how  $\nabla$  acts on vectors. The only two choices we have for vector 'multiplication' are the scalar product and the vector product. The scalar product gives the *divergence* of a vector defined as

$$\nabla \cdot \mathbf{A} \equiv \nabla_i A_i \equiv \frac{\partial A_i}{\partial x_i} \quad (2.30)$$

expanded as  $\nabla \cdot \mathbf{A} \equiv \nabla_i A_i = \nabla_1 A_1 + \nabla_2 A_2 + \nabla_3 A_3 = \nabla_x A_x + \nabla_y A_y + \nabla_z A_z$  or equivalently  $\nabla \cdot \mathbf{A} = \frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ .

The vector product gives the *curl* of a vector defined as

$$\nabla \times \mathbf{A} \equiv \epsilon_{ijk} (\nabla_i A_j) \hat{e}_k = \epsilon_{ijk} \frac{\partial A_j}{\partial x_i} \hat{e}_k \quad (2.31)$$

expanded as  $\nabla \times \mathbf{A} = (\frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3}) \hat{e}_1 + (\frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}) \hat{e}_2 + (\frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2}) \hat{e}_3$  or  $\nabla \times \mathbf{A} = (\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}) \hat{i} + (\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}) \hat{j} + (\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}) \hat{k}$ .

Thus the divergence, gradient and curl (often abbreviated as div, grad and curl) are the only ways of possibly combining  $\nabla$  with vectors and functions.

*Scalar and Vector Fields.* In order for equation (2.29) to make any sense it is obvious that the function must depend on the variables  $x, y, z$  as  $\phi = \phi(x, y, z)$ . This type of function is often called a *scalar field* because the value of the scalar is different at different points  $x, y, z$ . There is nothing complicated here. The scalar field  $\phi(x, y, z)$  is just the ordinary function of three variables that all calculus students know about. However the vector  $\mathbf{A}$  that we are talking about in equations (2.30) and (2.31) is *not* a fixed vector such as  $\mathbf{A} = 3\hat{i} + 2\hat{j} + 4\hat{k}$  that we saw in introductory physics. (It *could* be, but then  $\nabla \cdot \mathbf{A} = \nabla \times \mathbf{A} = 0$  which is the trivial case. We want to be more

general than this.) Rather  $\mathbf{A}$  is, in general, a *vector field*  $\mathbf{A} = \mathbf{A}(x, y, z)$  where each component of  $\mathbf{A}$  is itself a scalar field such as  $A_x(x, y, z)$ . Thus  $\mathbf{A} = \mathbf{A}(x, y, z) = A_x(x, y, z)\hat{i} + A_y(x, y, z)\hat{j} + A_z(x, y, z)\hat{k}$ . The careful reader will have already noticed that this must be the case if div and curl are to be non-zero. Note that whereas  $\mathbf{B} = 3\hat{i} + 2\hat{j} + 4\hat{k}$  is a *same* arrow at all points in space, the vector field  $\mathbf{A}(x, y, z)$  consists of a different arrow at all points in space. For example, suppose  $\mathbf{A}(x, y, z) = x^2y\hat{i} + z^2\hat{j} + xyz\hat{k}$ . Then  $\mathbf{A}(1, 1, 1) = \hat{i} + \hat{j} + \hat{k}$ , but  $\mathbf{A}(0, 1, 1) = \hat{j}$  etc.

**Example 2.5.1** Sketch a representative sample of vectors from the vector fields  $\mathbf{A}(x, y) = x\hat{i} + y\hat{j}$ ,  $\mathbf{B} = \hat{k}$  and  $\mathbf{C}(x, y) = -y\hat{i} + x\hat{j}$ . (These examples are from reference [2].)

**Solution**  $\mathbf{A}(x, y)$  is evaluated at a variety of points such as  $\mathbf{A}(0, 0) = 0$ ,  $\mathbf{A}(0, 1) = \hat{j}$ ,  $\mathbf{A}(1, 0) = \hat{i}$ ,  $\mathbf{A}(1, 1) = \hat{i} + \hat{j}$ ,  $\mathbf{A}(0, 2) = 2\hat{j}$  etc. We draw the corresponding arrow at each point to give the result shown in Fig. 2.1.

For the second case  $\mathbf{B} = \hat{j}$  the vector is independent of the coordinates  $(x, y, z)$ . This means that  $\mathbf{B} = \hat{j}$  at every point in space and is illustrated in Fig. 2.2.

Finally for  $\mathbf{C}(x, y)$  we have  $\mathbf{C}(1, 1) = -\hat{i} + \hat{j}$ ,  $\mathbf{C}(1, -1) = -\hat{i} - \hat{j}$ ,  $\mathbf{C}(-1, 1) = \hat{i} + \hat{j}$  etc. This is shown in Fig. 2.3.

Now let's study the very important physical implications of div, grad and curl [2, 5].

*Physical interpretation of Gradient.* We can most easily understand the meaning of  $\nabla\phi$  by considering the change in the function  $d\phi$  corresponding to a change in position  $d\mathbf{l}$  which is written  $d\mathbf{l} = dx\hat{i} + dy\hat{j} + dz\hat{k}$  [2, 8]. If  $\phi = \phi(x, y, z)$  then from elementary calculus it's change is given by  $d\phi = \frac{\partial\phi}{\partial x_i}dx_i = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz$ , which is nothing more than

$$d\phi = (\nabla\phi) \cdot d\mathbf{l} = |\nabla\phi||d\mathbf{l}|\cos\theta \quad (2.32)$$

Concerning the direction of  $\nabla\phi$ , it can be seen that  $d\phi$  will be a maximum when  $\cos\theta = 1$ , i.e. when  $d\mathbf{l}$  is chosen parallel to  $\nabla\phi$ . Thus if I move in the same direction as the gradient, then  $\phi$  changes maximally. Therefore the direction of  $\nabla\phi$  is along the greatest increase of  $\phi$ . (think of surfaces of constant  $\phi$  being given by contour lines on a map. Then the direction of the

2.5. DIV, GRAD AND CURL (DIFFERENTIAL CALCULUS FOR VECTORS)29

gradient is where the contour lines are closest together. ) The magnitude  $|\nabla\phi|$  gives the slope along this direction. (Each component of  $d\phi = \frac{\partial\phi}{\partial x_i} dx_i = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$  clearly gives a slope.)

The gradient of a function is most easily visualized for a two dimensional function  $\phi(x, y)$  where  $x$  and  $y$  are the latitude and longitude and  $\phi$  is the height of a hill. In this case surfaces of constant  $\phi$  will just be like the contour lines on a map. Given our discovery above that the direction of the gradient is in the direction of steepest ascent and the magnitude is the slope in this direction, then it is obvious that if we let a smooth rock roll down a hill then it will start to roll in the direction of the gradient with a speed proportional to the magnitude of the gradient. Thus the direction and magnitude of the gradient is the same as the direction and speed that a rock will take when rolling freely down a hill. If the gradient vanishes, then that means that you are standing on a local flat spot such as a summit or valley and a rock will remain stationary at that spot.

**Example 2.5.2** Consider the simple scalar field  $\phi(x, y) \equiv x$ . Compute the gradient and show that the magnitude and direction are consistent with the statements above.

**Solution** Clearly  $\phi$  only varies in the  $x$  direction and does not change in the  $y$  direction. Thus the direction of maximum increase is expected to be solely in the  $\hat{i}$  direction. This agrees with the computation of the gradient as  $\nabla\phi = \hat{i}$ . Every student knows that the straight line  $\phi = x$  has a slope of 1 which agrees with the computation of the magnitude as  $|\nabla\phi| = 1$ . (**do Problem 2.7**)

*Physical Interpretation of Divergence.* The divergence of a vector field represents the amount that the vector field *spreads out* or *diverges*. Consider our three examples of vector fields  $\mathbf{A}(x, y) = x\hat{i} + y\hat{j}$  and  $\mathbf{B} = \hat{j}$  and  $\mathbf{C}(x, y) = -y\hat{i} + x\hat{j}$ . From the pictures representing these vector fields (see Figs 2.1, 2.2 and 2.3) we can see that  $\mathbf{A}$  does spread out but  $\mathbf{B}$  and  $\mathbf{C}$  do not spread. Thus we expect that  $\mathbf{B}$  and  $\mathbf{C}$  have zero divergence. This is verified by calculating the divergence explicitly.

**Example 2.5.3** Calculate the divergence of  $\mathbf{A}(x, y) = x\hat{i} + y\hat{j}$

**Solution**  $\nabla \cdot \mathbf{A} \equiv \nabla_i A_i = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} = 1 + 1 = 2$

**Example 2.5.4** Calculate the divergence of  $\mathbf{B} = \hat{k}$ .

**Solution**  $\nabla \cdot \mathbf{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 + 0 + \frac{\partial}{\partial z}(1) = 0 + 0 + 0 = 0$  implying that the vector field does not spread out. **do Problem 2.8**

*Physical Interpretation of Curl* The curl measure the amount of rotation in a vector field. This could actually be measured by putting a little paddlewheel (with a fixed rotation axis) that is free to rotate in the vector field (like a little paddlewheel in a swirling water vortex). If the paddlewheel spins, then the vector field has a non-zero curl, but if the paddlewheel does not spin, then the vector field has zero curl. From Figs 2.1, 2.2 and 2.3 one expects that  $\mathbf{A}$  and  $\mathbf{B}$  would produce no rotation, but  $\mathbf{C}$  probably would. This is verified in the following examples.

**Example 2.5.5** Calculate the curl of  $\mathbf{A} = x\hat{i} + y\hat{j}$ .

**Solution**  $\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$   
 $= (0 - \frac{\partial y}{\partial z})\hat{i} + (\frac{\partial x}{\partial z} - 0)\hat{j} + (\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y})\hat{k} = (0 - 0)\hat{i} + (0 - 0)\hat{j} + (0 - 0)\hat{k} = 0$

**Example 2.5.6** Calculate the curl of  $\mathbf{C} = -y\hat{i} + x\hat{j}$ .

**Solution**  $\nabla \times \mathbf{C} = \left(\frac{\partial C_z}{\partial y} - \frac{\partial C_y}{\partial z}\right)\hat{i} + \left(\frac{\partial C_x}{\partial z} - \frac{\partial C_z}{\partial x}\right)\hat{j} + \left(\frac{\partial C_y}{\partial x} - \frac{\partial C_x}{\partial y}\right)\hat{k}$   
 $= (0 - \frac{\partial x}{\partial z})\hat{i} + (-\frac{\partial y}{\partial z} - 0)\hat{j} + (\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y})\hat{k} = (0 - 0)\hat{i} + (0 - 0)\hat{j} + (0 - 0)\hat{k} = 2\hat{k}$ .

Notice that the direction of the curl ( $\hat{k}$  in this case) is perpendicular to the plane of the vector field  $\mathbf{C}$ , as befits a vector product. **(do Problem 2.9).**

*Second Derivatives.* We have so far encountered the scalar  $\nabla \cdot \mathbf{A}$  and the vectors  $\nabla \phi$  and  $\nabla \times \mathbf{A}$ . We can operate again with  $\nabla$ . However we can only form the gradient of the scalar ( $\nabla \cdot \mathbf{A}$ ) as 1)  $\nabla(\nabla \cdot \mathbf{A})$ , but we can form the divergence and curl of both the vectors  $\nabla \phi$  and  $\nabla \times \mathbf{A}$  as 2)  $\nabla(\nabla \phi)$  and 3)  $\nabla \times (\nabla \phi)$  and 4)  $\nabla \cdot (\nabla \times \mathbf{A})$  and 5)  $\nabla \times (\nabla \times \mathbf{A})$ . However 3) and 4) are identically zero and 5) contains 1) and 2) already within it. (See [2] for discussions of this.)

The only second derivative that we shall use is  $\nabla(\nabla\phi) \equiv \nabla^2\phi$  often called the Laplacian of  $\phi$ , given by

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \quad (2.33)$$

*Exercise:* Verify that  $\nabla \times (\nabla\phi) = 0$  and  $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ .

## 2.6 Integrals of Div, Grad and Curl

In calculus courses [5] students will have learned about line, surface and volume integrals such as  $\int \mathbf{A} \cdot d\mathbf{l}$  or  $\int \mathbf{B} \cdot d\mathbf{A}$  or  $\int f d\tau$  where  $d\mathbf{l}$  is an oriented line segment,  $d\mathbf{A}$  is an oriented area element (which by definition always points perpendicular to the surface area) and  $d\tau$  is a volume element. We shall be interested in obtaining line, surface and volume integrals of div, grad and curl. One of the main uses of such integrals will be conversion of Maxwell's equations from differential to integral form.

In 3-dimensions we have 3 infinitesimal increments of length namely  $dx$ ,  $dy$  and  $dz$ . It is therefore very natural to put them together to form an infinitesimal length *vector* as

$$\begin{aligned} d\mathbf{l} &\equiv dl_x \hat{i} + dl_y \hat{j} + dl_z \hat{k} \\ &= dx \hat{i} + dy \hat{j} + dz \hat{k} \end{aligned} \quad (2.34)$$

It is also natural to combine them all into an infinitesimal volume element

$$d\tau \equiv dx dy dz \quad (2.35)$$

which is a *scalar*. How should we form an *area* element? Well we could either use  $dx dy$  or  $dx dz$  or  $dy dz$ . These 3 choices again suggest to us to form a *vector*

$$\begin{aligned} d\mathbf{A} &\equiv dA_x \hat{i} + dA_y \hat{j} + dA_z \hat{k} \\ &= dy dz \hat{i} + dx dz \hat{j} + dx dy \hat{k} \end{aligned} \quad (2.36)$$

where

$$\begin{aligned} dA_x &\equiv dy dz \\ dA_y &\equiv dx dz \\ dA_z &\equiv dx dy \end{aligned} \quad (2.37)$$



is the natural choice we would make. Note that, for example,  $dydz$  forms an area in the  $yz$  plane but *points* in the  $\hat{i}$  direction. Thus area is a vector that points in the direction perpendicular to the surface. Notice that if we had say 4 dimensions then we could form a 4-vector out of volume elements that would also have a direction in the 4-dimensional hyperspace.

We will be discussing four important results in this section namely the fundamental theorem of calculus, the fundamental theorem of gradients, the fundamental theorem of divergence (also called Gauss' theorem) and the fundamental theorem of curl (also called Stokes' theorem). However we will leave the proofs of these theorems to mathematics courses. Nevertheless we hope to make these theorems eminently believable via discussion and examples.

We proceed via analogy with the *fundamental theorem of calculus* which states

$$\int_a^b \frac{df}{dx} dx = f(b) - f(a) \quad (2.38)$$

where the derivative  $\frac{df}{dx}$  has been 'cancelled' out by the integral over  $dx$  to give a *right hand side that only depends on the end points of integration*.

The vector derivatives we have are  $\nabla f$ ,  $\nabla \cdot \mathbf{C}$  and  $\nabla \times \mathbf{C}$  and we wish to 'cancel' out the derivatives with an integral.

### 2.6.1 Fundamental Theorem of Gradients

The easiest to deal with is the gradient  $\nabla f \equiv \hat{i} \frac{\partial f}{\partial x} + \hat{j} \frac{\partial f}{\partial y} + \hat{k} \frac{\partial f}{\partial z}$  because its three components ( $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$ ,  $\frac{\partial f}{\partial z}$ ) are just ordinary (partial) derivatives like that appearing in the left hand side of the fundamental theorem of calculus equation (2.38). However, because  $\nabla f$  is a vector we can't just integrate over  $dx$  as in (2.38). We want to integrate  $\frac{\partial f}{\partial x}$  over  $dx$  and  $\frac{\partial f}{\partial y}$  over  $dy$  and  $\frac{\partial f}{\partial z}$  over  $dz$  and then we will have a three dimensional version of (2.38). The simplest way to do this is to integrate over  $d\mathbf{l} \equiv \hat{i} dx + \hat{j} dy + \hat{k} dz$  to give  $(\nabla f) \cdot d\mathbf{l} = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$  which is a three dimensional version of the integrand in (2.38). Griffiths [2] calls the result *the fundamental theorem of gradients*

$$\int (\nabla f) \cdot d\mathbf{l} = f(b) - f(a) \quad (2.39)$$

which is a result easily understood in light of (2.38). A point of note is that  $f = f(x)$  in (2.38), but  $f = f(x, y, z)$  in (2.39). Thus  $a$  and  $b$  in (2.39) actually represent a triple of coordinates. Note again that the right

hand side of (2.39) depends only on the endpoints. Three dimensional line integrals  $\int dl$  are different from one dimensional integrals  $\int dx$  in that three dimensional line integrals with the same end points can be performed over different paths as shown in Fig. 2.4, whereas the one dimensional integral always goes straight along the x axis.

Because the right hand side of (2.39) depends *only* on the end points of integration, there are two important corollaries of the fundamental theorem of gradients. The first is that

$$\int_a^b (\nabla f) \cdot d\mathbf{l} \text{ is independent of path of integration} \quad (2.40)$$

and secondly

$$\oint (\nabla f) \cdot d\mathbf{l} = 0 \quad (2.41)$$

where  $\oint d\mathbf{l}$  means that the integral has been performed around a closed loop with identical endpoints  $a = b$ . These two results follow automatically from our discussions above. Note that  $\int (\nabla f) \cdot d\mathbf{l}$  is independent of the path but this is not true for arbitrary vectors. In general  $\int \mathbf{C} \cdot d\mathbf{l}$  is not independent of path [2] (pg.31).

**Example 2.6.1** Let  $f = xy$  and  $a = (0, 0, 0)$  and  $b = (1, 1, 0)$ . Evaluate  $\int_a^b (\nabla f) \cdot d\mathbf{l}$  along two different integration paths and show that the results are the same.

**Solution** Let us choose the paths shown in Fig 2.5. First evaluate the integral along the  $AB$  path.

Path A ( $dy = dz = 0$ ):  $\int_A (\nabla f) \cdot d\mathbf{l} = \int_0^1 \frac{\partial f}{\partial x} dx = \int_0^1 y dx = 0$  because  $y = 0$  along this path.

Path B ( $dx = dz = 0$ ):  $\int_B (\nabla f) \cdot d\mathbf{l} = \int_0^1 \frac{\partial f}{\partial y} dy = \int_0^1 x dy = \int_0^1 dy = 1$  because  $x = 1$  along this path.

Thus  $\int_{AB} (\nabla f) \cdot d\mathbf{l} = \int_A (\nabla f) \cdot d\mathbf{l} + \int_B (\nabla f) \cdot d\mathbf{l} = 0 + 1 = 1$ .

Path C ( $dz = 0, y = x$ ):  $\int_C (\nabla f) \cdot d\mathbf{l} = \int (\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy) = \int y dx + \int x dy = 2 \int_0^1 x dx = 1$  because  $y = x$ . This answer is the same for the  $AB$  path. (**do Problem 2.10**).

**Example 2.6.2** Check the fundamental theorem of gradients using the above example.

**Solution** The above example has evaluated the left hand side of (2.39). To check the theorem we need to evaluate the right hand side for  $f = xy$ . Thus

$$f(a) = f(0, 0) = 0$$

$$f(b) = f(1, 1) = 1$$

$f(b) - f(a) = 1$  which agrees with the calculation in the previous example. (**do Problem 2.11**).

### 2.6.2 Gauss' theorem (Fundamental theorem of Divergence)

The divergence  $\nabla \cdot \mathbf{C}$  is a scalar quantity, so it is natural to expect that we integrate it with a scalar. Our integral elements are  $d\mathbf{l}$ ,  $d\mathbf{A}$  and  $d\tau$ , so we will be wanting  $\int (\nabla \cdot \mathbf{C}) d\tau$ . The *fundamental theorem of divergence* [2] (often also called Gauss' divergence theorem) is

$$\int (\nabla \cdot \mathbf{C}) d\tau = \oint \mathbf{C} \cdot d\mathbf{A} \equiv \Phi' \quad (2.42)$$

where  $\oint d\mathbf{A}$  denotes an integral over a *closed* surface and  $\Phi' \equiv \oint \mathbf{C} \cdot d\mathbf{A}$  is called the flux. Actually it is the flux over a *closed* surface. We shall denote  $\Phi \equiv \int d\mathbf{A}$  as the ordinary flux. It is easy to understand (2.42) in light of the fundamental theorem of gradients. Firstly  $d\tau$  can be thought of as  $d\tau = dx dy dz$  and so it will 'cancel' only say  $\frac{\partial}{\partial x}$  in  $\nabla \cdot \mathbf{C}$  and we will be left with  $dy dz$  which is the  $d\mathbf{A}$  integral on the right hand side of (2.42). We were unable to get rid of the entire  $d\tau$  integral because  $\nabla$  has only things like  $\frac{\partial}{\partial x}$  in it, which can only at most convert  $d\tau$  into  $d\mathbf{A}$ . Secondly the fact that we are left with a *closed* surface integral  $\oint d\mathbf{A}$  is exactly analogous to the *line* integral on the left hand side of (2.39) giving a term on the right hand side dependent only on the end *points*. A *closed* surface encloses a volume just as two end points enclose a line.

**Example 2.6.3** Check the divergence theorem for  $\mathbf{C} = x\hat{i} + yz\hat{j} + x\hat{k}$  using the volume given by the unit cube with one corner located at the origin.

**Solution**  $\nabla \cdot \mathbf{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z} = 1 + z + 0 = 1 + z$ .

The left hand side of (2.42) becomes

$$\int (\nabla \cdot \mathbf{C}) \cdot d\tau = \int_0^1 dx \int_0^1 dy \int_0^1 dz (1 + z) = \frac{3}{2} \int_0^1 dx \int_0^1 dy = \frac{3}{2}.$$

To get the right hand side we must evaluate the surface integral over all six sides of the cube (closed surface). The direction of each of these surfaces is shown in Fig. 2.6. The surface integral is  $\oint \mathbf{C} \cdot d\mathbf{A} = \int_A \mathbf{C} \cdot d\mathbf{A} + \int_B \mathbf{C} \cdot d\mathbf{A} + \int_C \mathbf{C} \cdot d\mathbf{A} + \int_D \mathbf{C} \cdot d\mathbf{A} + \int_E \mathbf{C} \cdot d\mathbf{A} + \int_F \mathbf{C} \cdot d\mathbf{A}$

For surface A ( $x = 1$ ):  $\mathbf{C} = \hat{i} + yz\hat{j} + \hat{k}$ ,  $d\mathbf{A} = dydz\hat{i}$ ,

giving  $\int_A \mathbf{C} \cdot d\mathbf{A} = \int_0^1 dy \int_0^1 dz = 1$ .

For surface B ( $y = 1$ ):  $\mathbf{C} = x\hat{i} + z\hat{j} + x\hat{k}$ ,  $d\mathbf{A} = dx dz \hat{j}$ ,

giving  $\int_B \mathbf{C} \cdot d\mathbf{A} = \int_0^1 dx \int_0^1 z dz = \frac{1}{2}$ .

For surface C ( $x = 0$ ):  $\mathbf{C} = yz\hat{j}$ ,  $d\mathbf{A} = -dydz\hat{i}$ ,

giving  $\int_C \mathbf{C} \cdot d\mathbf{A} = 0$ .

For surface D ( $y = 0$ ):  $\mathbf{C} = x\hat{i} + x\hat{k}$ ,  $d\mathbf{A} = -dx dz \hat{j}$ ,

giving  $\int_D \mathbf{C} \cdot d\mathbf{A} = 0$ .

For surface E ( $z = 0$ ):  $\mathbf{C} = x\hat{i} + x\hat{k}$ ,  $d\mathbf{A} = -dx dy \hat{k}$ ,

giving  $\int_E \mathbf{C} \cdot d\mathbf{A} = -\int_0^1 x dx \int_0^1 dy = -\frac{1}{2}$ .

For surface F ( $z = 1$ ):  $\mathbf{C} = x\hat{i} + y\hat{j} + x\hat{k}$ ,  $d\mathbf{A} = dx dy \hat{k}$ ,

giving  $\int_F \mathbf{C} \cdot d\mathbf{A} = \int_0^1 x dx \int_0^1 dy = \frac{1}{2}$ .

Thus  $\oint \mathbf{C} \cdot d\mathbf{A} = 1 + \frac{1}{2} + 0 + 0 - \frac{1}{2} + \frac{1}{2} = \frac{3}{2}$  in agreement with the left hand side. (**do Problem 2.12**).

### 2.6.3 Stokes' theorem (Fundamental theorem of curl)

Finally we integrate the vector  $\nabla \times \mathbf{C}$  with our remaining integral element of area  $d\mathbf{A}$ . The *fundamental theorem of curl* (often also called Stokes' theorem) is

$$\int (\nabla \times \mathbf{C}) \cdot d\mathbf{A} = \oint \mathbf{C} \cdot d\mathbf{l} \quad (2.43)$$

where  $\oint d\mathbf{l}$  denotes a line integral over a closed loop. To understand this result we apply similar reasoning as in the previous section. The  $\frac{\partial}{\partial x}$  in  $\nabla$

kills off  $dx$  in  $d\mathbf{A}$  leaving us with  $d\mathbf{l}$ . Also a closed loop encloses an area just as a closed area encloses a volume and end points enclose a line. Thus the right hand sides of (2.43) and (2.42) are analagous to the right hand side of (2.39).

As with the fundamental theorem of gradients, Stokes' theorem also has two corollaries, namely

$$\int(\nabla \times \mathbf{C}).d\mathbf{A} \text{ is independent of the surface of integration} \quad (2.44)$$

and

$$\oint(\nabla \times \mathbf{C}).d\mathbf{A} = 0 \quad (2.45)$$

which are both analagous to (2.40) and (2.41). One may ask why we didn't have two similar corollaries following Gauss' divergence theorem. The reason is that the area of (2.44) and (2.45) can be embedded in a volume and the line of (2.40) and (2.41) can be embedded in an area. Thus we can imagine different areas and lines to do our integrals. However if we confine ourselves to our three dimensional world then the volume is as high as we go. It is not embedded in anything else and so we don't have *different* volume integration paths to choose from. There is always only one. I suppose we could have said that  $\int(\nabla.\mathbf{C})d\tau$  is independent of the volume, but because all volume paths are identical, it is rather a useless statement.

## 2.7 Potential Theory

We shall discuss two important theorems from the mathematical subject known as potential theory [2, 8].

*Theorem 1.* If  $\mathbf{E}$  is a vector field, then  $\mathbf{E} = -\nabla V$  iff  $\nabla \times \mathbf{E} = 0$ , where  $V$  is a scalar field called the scalar potential.

*Theorem 2.* If  $\mathbf{B}$  is a vector field, then  $\mathbf{B} = \nabla \times \mathbf{A}$  iff  $\nabla.\mathbf{B} = 0$ , where  $\mathbf{A}$  is a vector field called the vector potential.

(The word iff is shorthand for 'if and only if'). The minus sign in Theorem 1 is arbitrary. It can simply be left off by writing  $V = -W$  where  $W$  is also a scalar potential. The only reason the minus sign appears is because physicists like to put it there.

So far these theorems have nothing to do with physics. They are just curious mathematical results at this stage. Also the 'if' part of the theorems

are easy to prove, but the 'only if' piece is difficult and won't be discussed here [7].

*Exercises:* If  $\mathbf{E} = -\nabla V$  show that  $\nabla \times \mathbf{E} = 0$ . If  $\mathbf{B} = \nabla \times \mathbf{A}$  show that  $\nabla \cdot \mathbf{B} = 0$ .

There are several corollaries that follow from the theorems above.

*Corollaries from Theorem 1:* i)  $\oint \mathbf{E} \cdot d\mathbf{l} = 0$  and ii)  $\oint \mathbf{E} \cdot d\mathbf{l}$  is independent of the integration path for given endpoints.

*Corollaries from Theorem 2:* i)  $\oint \mathbf{B} \cdot d\mathbf{A} = 0$  and ii)  $\oint \mathbf{B} \cdot d\mathbf{A}$  is independent of the integration surface for a given boundary line.

*Exercise:* Verify the above corollaries.

## 2.8 Curvilinear Coordinates

Before discussing curvilinear coordinates let us first review rectangular coordinates. Some references for this section are [2, 12, 13].

### 2.8.1 Plane Cartesian (Rectangular) Coordinates

For plane rectangular coordinates the  $\hat{i}$  and  $\hat{j}$  unit vectors lie along the  $x, y$  axes as shown on Fig. 2.7. Thus any vector  $\mathbf{A}$  is written as  $\mathbf{A} = A_x \hat{i} + A_y \hat{j}$  where  $(A_x, A_y)$  are the  $x$  and  $y$  components. Let  $dl_x$  and  $dl_y$  be infinitesimal increments of length obtained when moving in the  $\hat{i}$  and  $\hat{j}$  directions respectively. Obviously we then have

$$dl_x = dx \quad (2.46)$$

and

$$dl_y = dy \quad (2.47)$$

The infinitesimal displacement vector is then

$$d\mathbf{l} = dl_x \hat{i} + dl_y \hat{j} = dx \hat{i} + dy \hat{j}. \quad (2.48)$$

The infinitesimal area element (magnitude only) is

$$dA \equiv dl_x dl_y = dx dy. \quad (2.49)$$

in plane rectangular coordinates. Thus the area of a rectangle (for which plane rectangular coordinates are eminently appropriate) is given by Area of Rectangle =  $\int dA = \int_0^L dx \int_0^W dy = LW$ . That is, the area of a rectangle equals length times width.

### 2.8.2 Three dimensional Cartesian Coordinates

For rectangular coordinates the  $\hat{i}, \hat{j}, \hat{k}$  unit vectors lie along the  $x, y, z$  axes as shown in Fig. 2.8. Thus any vector is written  $\mathbf{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ . Let  $dl_x, dl_y, dl_z$  be infinitesimal increments of length obtained when moving in the  $\hat{i}, \hat{j}, \hat{k}$  directions respectively. Obviously we then have

$$dl_x = dx \quad (2.50)$$

$$dl_y = dy \quad (2.51)$$

$$dl_z = dz. \quad (2.52)$$

The infinitesimal volume element is

$$d\tau \equiv dl_x dl_y dl_z = dx dy dz. \quad (2.53)$$

in plane rectangular coordinates. Thus for example the volume of a cube (for which plane rectangular coordinates are eminently appropriate) is given by Volume of Cube =  $\int d\tau = \int_0^L dx \int_0^W dy \int_0^H dz = LWH$ . That is, the volume of a cube equals length times width times height.

The reason for going through this analysis with rectangular coordinates is to shed light on the results for curvilinear coordinates to which we now turn.

### 2.8.3 Plane (2-dimensional) Polar Coordinates

Before discussing these coordinates let us first consider how the coordinates of a point  $P(x, y)$  (or the components of a vector  $\mathbf{A}$ ) change to  $P(x', y')$  when the axis system is rotated by angle  $\theta$  as shown in Fig. 2.10. Upon examination of the figure we have

$$\begin{aligned} x' &= x \cos \theta + y \sin \theta \\ y' &= y \cos \theta - x \sin \theta \end{aligned} \quad (2.54)$$

or

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (2.55)$$

or, inverted as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \quad (2.56)$$

Note that a *counterclockwise rotation is defined to be positive*. These results will be used extensively. Let us now discuss plane polar coordinates.

This 2-dimensional coordinate system is specified by radial ( $r$ ) and angular ( $\theta$ ) coordinates together with unit vectors ( $\hat{e}_r, \hat{e}_\theta$ ) shown in Fig. 2.9. ( $\theta$  varies from 0 to  $2\pi$ ). Whereas for rectangular coordinates the basis vectors are fixed in position, now with plane polar coordinates the basis vectors are attached to the moving point  $P$  and  $\hat{e}_r, \hat{e}_\theta$  move as  $P$  moves. The relation between plane polar and plane rectangular coordinates is obviously

$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta\end{aligned}\tag{2.57}$$

or

$$\begin{aligned}r^2 &= x^2 + y^2 \\ \theta &= \arctan(y/x)\end{aligned}\tag{2.58}$$

Just as a vector  $\mathbf{A}$  is written in rectangular coordinates as  $\mathbf{A} = A_x \hat{i} + A_y \hat{j}$ , so too in polar coordinates we have

$$\mathbf{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta\tag{2.59}$$

where ( $A_r, A_\theta$ ) are the radial and angular coordinates respectively. Clearly  $A_r$  and  $A_\theta$  can be thought of as the  $x', y'$  coordinates introduced above.

Thus

$$\begin{pmatrix} A_r \\ A_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} A_x \\ A_y \end{pmatrix}\tag{2.60}$$

and so  $\mathbf{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta = (A_x \cos \theta + A_y \sin \theta) \hat{e}_r + (-A_x \sin \theta + A_y \cos \theta) \hat{e}_\theta = A_x \hat{i} + A_y \hat{j}$ . Equating coefficients of  $A_x$  and  $A_y$  respectively, we have  $\hat{i} = \cos \theta \hat{e}_r - \sin \theta \hat{e}_\theta$  and  $\hat{j} = \sin \theta \hat{e}_r + \cos \theta \hat{e}_\theta$  or

$$\begin{pmatrix} \hat{i} \\ \hat{j} \end{pmatrix} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{e}_r \\ \hat{e}_\theta \end{pmatrix}\tag{2.61}$$

or

$$\begin{pmatrix} \hat{e}_r \\ \hat{e}_\theta \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \end{pmatrix}.\tag{2.62}$$

That is, the relation between the unit vectors is

$$\begin{aligned}\hat{e}_r &= \cos \theta \hat{i} + \sin \theta \hat{j} \\ \hat{e}_\theta &= -\sin \theta \hat{i} + \cos \theta \hat{j}.\end{aligned}\tag{2.63}$$



Let  $dl_r$  and  $dl_\theta$  be infinitesimal increments of length obtained when moving in the  $\hat{e}_r$  and  $\hat{e}_\theta$  directions respectively. From Fig. 2.9 we have

$$dl_r = dr \quad (2.64)$$

and

$$dl_\theta = r d\theta \quad (2.65)$$

as shown in Fig. 2.11. The infinitesimal displacement vector is then

$$d\mathbf{l} = dl_r \hat{e}_r + dl_\theta \hat{e}_\theta = dr \hat{e}_r + r d\theta \hat{e}_\theta \quad (2.66)$$

The circumference of a circle (for which plane polar coordinates are eminently appropriate) is given by  $\int dl_\theta = \int_0^{2\pi} r d\theta = 2\pi r$ . The infinitesimal area element (magnitude only) is

$$dA \equiv dl_r dl_\theta = r dr d\theta \quad (2.67)$$

so that the area of a circle is  $\int dA = \int dl_r dl_\theta = \int_0^R r dr \int_0^{2\pi} d\theta = (\frac{1}{2}R^2)(2\pi) = \pi R^2$ .

### 2.8.4 Spherical (3-dimensional) Polar Coordinates

This coordinate system is specified by  $(r, \theta, \phi)$  and unit vectors  $(\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi)$  shown in Fig. 2.12. In order to sweep out a sphere  $\phi$  varies from 0 to  $2\pi$ , but  $\theta$  now only varies from 0 to  $\pi$ . (Note that  $\hat{e}_\theta$  points 'down' whereas for plane polar coordinates it pointed 'up'.) Again the basis vectors move as point  $P$  moves. The relation between polar and rectangular coordinates is obtained from Fig. 2.13 as

$$\begin{aligned} x &= r \sin \theta \cos \phi \\ y &= r \sin \theta \sin \phi \\ z &= r \cos \theta \end{aligned} \quad (2.68)$$

where  $r^2 = x^2 + y^2 + z^2$ . The unit vectors are related by

$$\begin{aligned} \hat{e}_r &= \sin \theta \cos \phi \hat{i} + \sin \theta \sin \phi \hat{j} + \cos \theta \hat{k} \\ \hat{e}_\theta &= \cos \theta \cos \phi \hat{i} + \cos \theta \sin \phi \hat{j} - \sin \theta \hat{k} \\ \hat{e}_\phi &= -\sin \phi \hat{i} + \cos \phi \hat{j} \end{aligned} \quad (2.69)$$

or

$$\begin{pmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{pmatrix} = \begin{pmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\ -\sin \phi & \cos \phi & 0 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}. \quad (2.70)$$

**(do Problem 2.13)** Any vector  $\mathbf{A}$  is written as  $\mathbf{A} = A_r \hat{e}_r + A_\theta \hat{e}_\theta + A_\phi \hat{e}_\phi$ . Let  $dl_r, dl_\theta, dl_\phi$  be infinitesimal increments of length obtained when moving in the  $\hat{e}_r, \hat{e}_\theta$  and  $\hat{e}_\phi$  directions respectively. Clearly

$$dl_r = dr \quad (2.71)$$

and

$$dl_\theta = r d\theta \quad (2.72)$$

but

$$dl_\phi = r \sin \theta d\phi \quad (2.73)$$

which can be seen from Fig. 2.14. The infinitesimal displacement vector is then

$$d\mathbf{l} = dl_r \hat{e}_r + dl_\theta \hat{e}_\theta + dl_\phi \hat{e}_\phi = dr \hat{e}_r + r d\theta \hat{e}_\theta + r \sin \theta d\phi \hat{e}_\phi \quad (2.74)$$

There are three infinitesimal area elements that we can form, namely  $dA' = dl_r dl_\theta = r dr d\theta$  (variable  $r$ ) or  $dA'' = dl_r dl_\phi = r^2 \sin \theta dr d\phi$  (variable  $r$ ) and the one which has fixed  $r$  and gives the area patch on the surface of a sphere

$$dA = dl_\theta dl_\phi = r^2 \sin \theta d\theta d\phi. \quad (2.75)$$

Thus the surface area of a sphere is  $\int dA = \int dl_\theta dl_\phi = R^2 \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi = 4\pi R^2$ . The infinitesimal volume element is

$$d\tau = dl_r dl_\theta dl_\phi = r^2 \sin \theta dr d\theta d\phi. \quad (2.76)$$

so that the volume of a sphere is  $\int d\tau = \int dl_r dl_\theta dl_\phi = \int_0^R r^2 dr \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi = \frac{4}{3}\pi R^3$ .

### 2.8.5 Cylindrical (3-dimensional) Polar Coordinates

These coordinates specified by  $(\rho, \phi, z)$  and  $(\hat{e}_\rho, \hat{e}_\phi, \hat{k})$  are shown in Fig 2.15. It is worthwhile to note that for a fixed slice in  $z$ , cylindrical polar coordinates are identical to plane polar coordinates. The angle  $\theta$  and radius  $r$  used in plane polar coordinates is replaced by angle  $\phi$  and radius  $\rho$  in cylindrical

polar coordinates. In other words cylindrical polar coordinates are just plane polar coordinates with a  $z$  axis tacked on. The relation between cylindrical polar coordinates and rectangular coordinates is thus

$$\begin{aligned}x &= \rho \cos \phi \\y &= \rho \sin \phi \\z &= z\end{aligned}\tag{2.77}$$

the first two of which are analogous to equation (2.57). Unit vectors are related by

$$\begin{aligned}\hat{e}_\rho &= \cos \phi \hat{i} + \sin \phi \hat{j} \\ \hat{e}_\phi &= -\sin \phi \hat{i} + \cos \phi \hat{j} \\ \hat{e}_z &= \hat{k}\end{aligned}\tag{2.78}$$

again the first two of which are just equation (2.63). Any vector is

$$\mathbf{A} = A_\rho \hat{e}_\rho + A_\phi \hat{e}_\phi + A_z \hat{e}_z.\tag{2.79}$$

Note that for spherical polar coordinates the position vector of point P is  $\mathbf{r} = r\hat{e}_r$  and for plane polar coordinates  $\mathbf{r} = r\hat{e}_r$  also. However in cylindrical polar coordinates we have

$$\mathbf{r} = \rho \hat{e}_\rho + z \hat{e}_z\tag{2.80}$$

as can be seen from Fig. 2.15. The infinitesimal elements of length are

$$dl_\rho = d\rho\tag{2.81}$$

$$dl_\phi = \rho d\phi\tag{2.82}$$

and

$$dl_z = dz\tag{2.83}$$

where (2.81) and (2.82) are the same as (2.64) and (2.65). The infinitesimal displacement vector is

$$d\mathbf{l} = dl_\rho \hat{e}_\rho + dl_\phi \hat{e}_\phi + dl_z \hat{e}_z = d\rho \hat{e}_\rho + \rho d\phi \hat{e}_\phi + dz \hat{e}_z\tag{2.84}$$

to be compared to (2.66). *Exercise:* derive the formula for the volume of a cylinder.

### 2.8.6 Div, Grad and Curl in Curvilinear Coordinates

See Griffiths.

(to be written up later)

## 2.9 Summary

## 2.10 Problems

**2.1 a)** Vector  $\mathbf{C}$  points out of the page and  $\mathbf{D}$  points to the right in the same plane. What is the direction of  $\mathbf{C} \times \mathbf{D}$  and  $\mathbf{D} \times \mathbf{C}$  ?

**b)**  $\mathbf{B}$  points to the left and  $\mathbf{A}$  points down the page. What is the direction of  $\mathbf{B} \times \mathbf{A}$  and  $\mathbf{A} \times \mathbf{B}$  ?

**2.2 a)** Write down the values of the following Kronecker delta symbols:  $\delta_{11}, \delta_{12}, \delta_{33}, \delta_{13}$ .

**b)** Write down the values of the following Levi-Civita symbols:  $\epsilon_{111}, \epsilon_{121}, \epsilon_{312}, \epsilon_{132}$ .

**2.3** Show that  $(\mathbf{A} \times \mathbf{B})_z = A_x B_y - A_y B_x$ .

**2.4** Show that the determinant formula (2.23) gives the same results as (2.22).

**2.5** Show that (2.24) is true for the following values of indices:

**a)**  $i = 1, j = 1, l = 1, m = 1$ ,

**b)**  $i = 3, j = 1, l = 1, m = 3$ .

**2.6** Prove the following vector identities:

**a)**  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$

**b)**  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C})$

**c)**  $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$

**d)**  $(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) = (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C})$

**e)**  $\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = \mathbf{B}[\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})] - (\mathbf{A} \cdot \mathbf{B})(\mathbf{C} \times \mathbf{D})$ .

**2.7** Calculate the gradient of  $f(x, y, z) = x + yz^2$ .

**2.8** Calculate the divergence of  $\mathbf{C} = -y\hat{i} + x\hat{j}$  and interpret your result.

**2.9** Calculate the curl of  $\mathbf{B} = \hat{j}$  and interpret your result.

**2.10** Let  $f = xy^2$  and  $a = (0, 0, 0)$  and  $b = (1, 1, 0)$ . Evaluate  $\int_a^b \nabla f \cdot d\mathbf{l}$  along two different integration paths and show that the results are the same.

**2.11** Check the fundamental theorem of gradients using the function and end points of Problem 2.10.

**2.12** Check Gauss' divergence theorem using  $\mathbf{C} = xz\hat{i} + y^2\hat{j} + yz\hat{k}$  using the unit cube with a corner at the origin as shown in Fig. 2.6.

**2.13** Prove equation (2.69) or (2.70).

**2.11** Answers

**2.1 a)**  $\mathbf{C} \times \mathbf{D}$  points up the page and  $\mathbf{D} \times \mathbf{C}$  points down the page.

**b)**  $\mathbf{B} \times \mathbf{A}$  points out of the page and  $\mathbf{A} \times \mathbf{B}$  points into the page.

**2.2 a)** 1, 0, 1, 0.

**b)** 0, 0, +1, -1.

**2.7**  $\hat{i} + z^2\hat{j} + 2yz\hat{k}$ .

**2.8** 0

**2.9** 0

**2.10** 1

## 2.12 Solutions

**2.1 a)**  $\mathbf{C} \times \mathbf{D}$  points up the page and  $\mathbf{D} \times \mathbf{C}$  points down the page.

**b)**  $\mathbf{B} \times \mathbf{A}$  points out of the page and  $\mathbf{A} \times \mathbf{B}$  points into the page.

**2.2 a)** 1, 0, 1, 0.

**b)** 0, 0, +1, -1.

**2.3** From (2.20)  $\mathbf{A} \times \mathbf{B} = \epsilon_{ijk} A_i B_j \hat{e}_k$ . Therefore the  $k$ th component is  $(\mathbf{A} \times \mathbf{B})_k = \epsilon_{ijk} A_i B_j$ . Thus  $(\mathbf{A} \times \mathbf{B})_3 = \epsilon_{ij3} A_i B_j = \epsilon_{1j3} A_1 B_j + \epsilon_{2j3} A_2 B_j + \epsilon_{3j3} A_3 B_j$ . Now  $\epsilon_{3j3} = 0$  for all values of  $j$  and in the first two terms the only non-zero values will be  $j = 2$  and  $j = 1$  respectively. Realizing this saves us from writing out all the terms in the sum over  $j$ . Thus  $(\mathbf{A} \times \mathbf{B})_3 = \epsilon_{123} A_1 B_2 + \epsilon_{213} A_2 B_1 = A_1 B_2 - A_2 B_1$  meaning that  $(\mathbf{A} \times \mathbf{B})_z = A_x B_y - A_y B_x$ .

### 2.4

$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix} \\ &= \hat{e}_1 A_2 B_3 + \hat{e}_2 A_3 B_1 + \hat{e}_3 A_1 B_2 \\ &\quad - \hat{e}_3 A_1 B_2 - \hat{e}_2 A_1 B_3 - \hat{e}_1 A_3 B_2 \\ &= (A_2 B_3 - A_3 B_2) \hat{e}_1 + (A_3 B_1 - A_1 B_3) \hat{e}_2 \\ &\quad + (A_1 B_2 - A_2 B_1) \hat{e}_3. \end{aligned}$$



$$\mathbf{2.5} \quad \epsilon_{kij}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$$

a) The left hand side is

$$\begin{aligned} \epsilon_{kij}\epsilon_{klm} &= \epsilon_{k11}\epsilon_{k11} \\ &= \epsilon_{111}\epsilon_{111} + \epsilon_{211}\epsilon_{211} + \epsilon_{311}\epsilon_{311} \\ &= (0)(0) + (0)(0) + (0)(0) \\ &= 0. \end{aligned}$$

The right hand side is  $\delta_{11}\delta_{11} - \delta_{11}\delta_{11} = (1)(1) - (1)(1) = 1 - 1 = 0$

b) The left hand side is

$$\begin{aligned} \epsilon_{kij}\epsilon_{klm} &= \epsilon_{k31}\epsilon_{k13} \\ &= \epsilon_{131}\epsilon_{113} + \epsilon_{231}\epsilon_{213} + \epsilon_{331}\epsilon_{313} \\ &= (0)(0) + (+1)(-1) + (0)(0) \\ &= 0 - 1 + 0 = -1. \end{aligned}$$

The right hand side is  $\delta_{31}\delta_{13} - \delta_{33}\delta_{11} = (0)(0) - (1)(1) = -1$

## 2.6

a)

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= A_k (\mathbf{B} \times \mathbf{C})_k = A_k \epsilon_{ijk} B_i C_j \\ &= B_i \epsilon_{ijk} C_j A_k = B_i \epsilon_{jki} C_j A_k = B_i (\mathbf{C} \times \mathbf{A})_i = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})\end{aligned}$$

Also this is

$$= C_j \epsilon_{ijk} A_k B_i = C_j \epsilon_{kij} A_k B_i = C_j (\mathbf{A} \times \mathbf{B})_j = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}).$$

b)

$$\begin{aligned}\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) &= A_k (\mathbf{B} \times \mathbf{C})_k = A_k \epsilon_{ijk} B_i C_j \\ &= B_i \epsilon_{ijk} A_k C_j = B_i \epsilon_{jki} A_k C_j = -B_i \epsilon_{kji} A_k C_j \\ &= -B_i (\mathbf{A} \times \mathbf{C})_i = -\mathbf{B} \cdot (\mathbf{A} \times \mathbf{C}).\end{aligned}$$

c)

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \epsilon_{ijk} A_i (\mathbf{B} \times \mathbf{C})_j \hat{e}_k = \epsilon_{ijk} A_i \epsilon_{jlm} B_l C_m \hat{e}_k \\ &= \epsilon_{ijk} \epsilon_{jlm} A_i B_l C_m \hat{e}_k = -\epsilon_{ikj} \epsilon_{jlm} A_i B_l C_m \hat{e}_k \\ &= -(\delta_{il} \delta_{km} - \delta_{im} \delta_{kl}) A_i B_l C_m \hat{e}_k \\ &= -A_i B_i C_k \hat{e}_k + A_i B_k C_i \hat{e}_k \\ &= -\mathbf{C}(\mathbf{A} \cdot \mathbf{B}) + \mathbf{B}(\mathbf{A} \cdot \mathbf{C}).\end{aligned}$$

d)

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \cdot (\mathbf{C} \times \mathbf{D}) &= (\mathbf{A} \times \mathbf{B})_k (\mathbf{C} \times \mathbf{D})_k \\ &= \epsilon_{ijk} A_i B_j \epsilon_{lmk} C_l D_m \\ &= \epsilon_{ijk} \epsilon_{lmk} A_i B_j C_l D_m \\ &= \epsilon_{kij} \epsilon_{klm} A_i B_j C_l D_m \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_i B_j C_l D_m \\ &= A_i B_j C_i D_j - A_i B_j C_j D_i \\ &= (\mathbf{A} \cdot \mathbf{C})(\mathbf{B} \cdot \mathbf{D}) - (\mathbf{A} \cdot \mathbf{D})(\mathbf{B} \cdot \mathbf{C}).\end{aligned}$$

e)

$$\mathbf{A} \times [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})] = \epsilon_{ijk} A_i [\mathbf{B} \times (\mathbf{C} \times \mathbf{D})]_j \hat{e}_k$$

$$\begin{aligned}
&= \epsilon_{ijk} A_i \epsilon_{lmj} B_l (\mathbf{C} \times \mathbf{D})_m \hat{e}_k = \epsilon_{ijk} A_i \epsilon_{lmj} B_l \epsilon_{stm} C_s D_t \hat{e}_k \\
&= \epsilon_{ijk} \epsilon_{lmj} A_i B_l \epsilon_{stm} C_s D_t \hat{e}_k = \epsilon_{jki} \epsilon_{jlm} A_i B_l \epsilon_{stm} C_s D_t \hat{e}_k \\
&\quad = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) A_i B_l \epsilon_{stm} C_s D_t \hat{e}_k \\
&= A_i B_k \epsilon_{sti} C_s D_t \hat{e}_k - A_i B_i \epsilon_{stk} C_s D_t \hat{e}_k \\
&\quad = \mathbf{B} A_i \epsilon_{sti} C_s D_t - (\mathbf{A} \cdot \mathbf{B}) \epsilon_{stk} C_s D_t \hat{e}_k \\
&\quad = \mathbf{B} A_i (\mathbf{C} \times \mathbf{D})_i - (\mathbf{A} \cdot \mathbf{B}) (\mathbf{C} \times \mathbf{D}) \\
&\quad = \mathbf{B} [\mathbf{A} \cdot (\mathbf{C} \times \mathbf{D})] - (\mathbf{A} \cdot \mathbf{B}) (\mathbf{C} \times \mathbf{D}).
\end{aligned}$$

## 2.13 Figure captions for chapter 2

Fig. 2.1 Sketch of the vector field  $\mathbf{A}(x, y) = x\hat{i} + y\hat{j}$ .

Fig. 2.2 Sketch of the vector field  $\mathbf{B} = \hat{j}$ .

Fig. 2.3 Sketch of the vector field  $\mathbf{C}(x, y) = -y\hat{i} + x\hat{j}$ .

Fig. 2.4 A multidimensional line integral with the same end points  $(a, b)$  can be performed over different paths.

Fig. 2.5 Integration paths used in Example 2.6.1.

Fig. 2.6 Integration areas used in Example 2.6.3.

Fig. 2.7 Plane (Rectangular) Coordinates and Basis Vectors.

Fig. 2.8 3-dimensional Cartesian Coordinates and Basis Vectors.

Fig. 2.9 Plane Polar Coordinates and Basis Vectors.

Fig. 2.10 Rotation of a coordinate system.

Fig. 2.11 An increment of length  $dl_\theta$  moving in the  $\hat{e}_\theta$  direction.

Fig. 2.12 Spherical Polar Coordinates and Basis Vectors.

Fig. 2.13 Relating Spherical Polar Coordinates to Cartesian Coordinates.

Fig. 2.14 An increment of length  $dl_\phi$  moving in the  $\hat{e}_\phi$  direction.

Fig. 2.15 Cylindrical Polar Coordinates and Basis Vectors. (Note that the coordinates and basis vectors in the  $x, y$  plane are identical to the plane polar coordinates and basis vectors of Fig. 2.9.)



## Chapter 3

# MAXWELL'S EQUATIONS

This is a book about the subject of classical electrodynamics. The word *classical* as used in physics does not mean 'old' or 'pre-twentieth century' or 'non-relativistic' (as many students think). The word classical is very specific and simply means that the theory is *non quantum-mechanical*. Actually classical physics is still an active area of research today particularly those branches dealing with such topics as fluid dynamics and chaos [14]. In fact one of the major unsolved problems in classical physics is the theory of turbulence (reference ?).

Classical electrodynamics is based entirely on Maxwell's equations. In fact one could define classical electrodynamics as the study of Maxwell's equations. Contrast this to the theory of Quantum Electrodynamics which results when one considers the quantization of the electromagnetic field [15]. Maxwell's equations are a set of four fundamental equations that cannot be derived from anywhere else. They are really a guess as to how nature behaves. The way we check whether they are correct is by comparing their predictions to experiment. They are analagous to say Newton's laws in classical mechanics, or Schrodinger's equation in quantum mechanics, or the Einstein field equations in general relativity, all of which are not derivable from anywhere else but are considered the starting postulates for the theory. All we can do is to write down the postulated equation and calculate the consequences. Of course it is always fascinating to study how these equations were originally guessed at and what experiments or concepts lead to them being proposed [16].

The theory of classical electrodynamics was developed in piecemeal fashion [16, 9] with the establishment of Coulomb's law, the Biot-Savart law,

Gauss' law, Faraday's law, Ampère's law etc. However it was James Clerk Maxwell, who in 1861, unified all the previous work, made some significant corrections of his own, and finally wrote down what are today called Maxwell's equations.

### 3.1 Maxwell's equations in differential form

Maxwell's equations (in *vacuum*) consist of Gauss' law for the electric field  $\mathbf{E}$ ,

$$\nabla \cdot \mathbf{E} = 4\pi k\rho, \quad (3.1)$$

Gauss' law for the magnetic field  $\mathbf{B}$ ,

$$\nabla \cdot \mathbf{B} = 0, \quad (3.2)$$

Faraday's law

$$\nabla \times \mathbf{E} + g \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (3.3)$$

and Ampère's law

$$\nabla \times \mathbf{B} - \frac{1}{gc^2} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi k}{gc^2} \mathbf{j} \quad (3.4)$$

where  $\rho$  is the charge density (charge per unit volume  $\rho \equiv \frac{dq}{d\tau}$ ) and  $\mathbf{j}$  is the current density (charge per unit area  $\mathbf{j} \equiv \frac{di}{dA} \hat{\mathbf{A}}$ ).  $k$  and  $g$  are constants and  $c$  is the speed of light in vacuum.

The Lorentz force law is

$$\mathbf{F} = q(\mathbf{E} + g\mathbf{v} \times \mathbf{B}) \quad (3.5)$$

which gives the force  $\mathbf{F}$  on a particle of charge  $q$  moving with velocity  $\mathbf{v}$  in an electromagnetic field.

Later we shall see that the constant  $k$  is the same one that appears in Coulomb's law for the electric force between two point charges

$$\mathbf{F} = k \frac{q_1 q_2}{r^2} \hat{\mathbf{r}} \quad (3.6)$$

and the constant  $g$  specifies the relative strength of the  $\mathbf{E}$  and  $\mathbf{B}$  fields. An excellent and more complete discussion of units may be found in the book by Jackson [10]. In terms of Jackson's constants ( $k_1$  and  $k_3$ ) the relation is  $k = k_1$  and  $g = k_3$ . From Coulomb's law it can be seen that the units

chosen for charge and length will determine the units for  $k$ . The three main systems of units in use are called Heaviside-Lorentz, CGS or Gaussian and MKS or SI. The values of the constants in these unit systems are specified in Table 3.1 below.

	Heaviside-Lorentz	CGS (Gaussian)	SI (MKS)
$k$	$\frac{1}{4\pi}$	1	$\frac{1}{4\pi\epsilon_0}$
$g$	$\frac{1}{c}$	$\frac{1}{c}$	1

Table 3.1

Inserting these constants into Maxwell's equations (3.1) - (3.4) and the Lorentz force law gives the equations as they appear in different unit systems. In **Heaviside-Lorentz units** Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \rho \quad (3.7)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3.8)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (3.9)$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c} \mathbf{j} \quad (3.10)$$

and the Lorentz force law is

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{B}). \quad (3.11)$$

In **CGS or Gaussian units** Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 4\pi\rho \quad (3.12)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3.13)$$

$$\nabla \times \mathbf{E} + \frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (3.14)$$

$$\nabla \times \mathbf{B} - \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c} \mathbf{j} \quad (3.15)$$



and the Lorentz force law is

$$\mathbf{F} = q(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}). \quad (3.16)$$

In **MKS** or **SI units** Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (3.17)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (3.18)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad (3.19)$$

$$\nabla \times \mathbf{B} - \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2 \epsilon_0} \mathbf{j} \quad (3.20)$$

However, because  $c^2 = \frac{1}{\mu_0 \epsilon_0}$  (see Section x.x) in SI units this last equation is usually written

$$\nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} \quad (3.21)$$

and the Lorentz force law is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \quad (3.22)$$

Particle physicists [17] most often use Heaviside-Lorentz units and furthermore usually use units in which  $c \equiv 1$ , so that Maxwell's equations are in their simplest possible form. CGS units are used in the books by Jackson [10] and Ohanian [9] and Marion [11], whereas MKS units are used by Griffiths [2] and most freshman physics texts. In this book we shall use Maxwell's equations as presented in equations (3.1) - (3.4) so that all of our equations will contain the constants  $k$  and  $n$ . The equations for a specific unit system can then simply be obtained by use of Table 3.1. Thus this book will *not* make a specific choice of units. The advantage of this is that comparison of results in this book with the references will be made much easier.

### 3.2 Maxwell's equations in integral form

In freshman physics course one usually does not study Maxwell's equations in differential form but rather one studies them in integral form. Let us now prove that the Maxwell's equations as presented in equations (3.1) - (3.4) do in fact give the equations that one has already studied in freshman physics.

To accomplish this we perform a volume integral over the first two equations and an area integral over the second two equations.

Integrating over volume ( $\int d\tau$ ) on Gauss' law for  $\mathbf{E}$  (3.1) and using Gauss' divergence theorem as in  $\int(\nabla\cdot\mathbf{E})d\tau = \oint\mathbf{E}\cdot d\mathbf{A}$ , and  $q = \int\rho d\tau$  yields

$$\Phi'_E \equiv \oint\mathbf{E}\cdot d\mathbf{A} = 4\pi kq \quad (3.23)$$

which is Gauss' law for the electric flux  $\Phi'_E$  over a closed surface area. The magnetic equation (3.2) similarly becomes

$$\Phi'_B \equiv \oint\mathbf{B}\cdot d\mathbf{A} = 0 \quad (3.24)$$

where  $\Phi'_B$  is the magnetic flux over a closed surface area.

Integrating over area ( $\int d\mathbf{A}$ ) on Faraday's law and using Stokes' curl theorem, as in  $\int(\nabla\times\mathbf{E})\cdot d\mathbf{A} = \oint\mathbf{E}\cdot d\mathbf{l}$ , yields

$$\oint\mathbf{E}\cdot d\mathbf{l} + g\frac{\partial\Phi_B}{\partial t} = 0 \quad (3.25)$$

where  $\Phi_B \equiv \oint\mathbf{B}\cdot d\mathbf{A}$  is the magnetic flux (*not* necessarily over a closed surface area). Finally integrating Ampère's law over an area and using  $i \equiv \int\mathbf{j}\cdot\mathbf{A}$ , yields

$$\oint\mathbf{B}\cdot d\mathbf{l} - \frac{1}{gc^2}\frac{\partial\Phi_E}{\partial t} = \frac{4\pi k}{gc^2}i \quad (3.26)$$

where  $\Phi_E \equiv \oint\mathbf{E}\cdot d\mathbf{A}$  is the electric flux (not necessarily over a closed surface area).

This completes our derivation of Maxwell's equations in integral form.

*Exercise:* Using Table 3.1, check that the equations above are the equations that you studied in freshman physics.

### 3.3 Charge Conservation

Conservation of charge is implied by Maxwell's equations. Taking the divergence of Ampère's law gives  $\nabla\cdot(\nabla\times\mathbf{B}) - \frac{1}{gc^2}\frac{\partial}{\partial t}\nabla\cdot\mathbf{E} = \frac{4\pi k}{gc^2}\nabla\cdot\mathbf{j}$ . However  $\nabla\cdot(\nabla\times\mathbf{B}) = 0$  (**do Problem 3.1**) and using the electric Gauss law we have  $-\frac{4\pi k}{gc^2}\frac{\partial\rho}{\partial t} = \frac{4\pi k}{gc^2}\nabla\cdot\mathbf{j}$ . We see that the constants cancel leaving

$$\nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t} = 0 \quad (3.27)$$

which is the *continuity equation*, or conservation of charge in differential form. Because the constants all cancelled, this equation is the same in all systems of units. This equation is a *local* conservation law in that it tells us how charge is conserved locally. That is if the charge density increases in some region locally (yielding a non-zero  $\frac{\partial \rho}{\partial t}$ ), then this is caused by current flowing into the local region by the amount  $\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{j}$ . If the charge decreases in a local region (a negative  $-\frac{\partial \rho}{\partial t}$ ), then this is due to current flowing out of that region by the amount  $-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{j}$ . The divergence here is positive corresponding to  $\mathbf{j}$  spreading outwards (see Fig. 2.1).

Contrast this with the *global* conservation law obtained by integrating over the volume of the whole global universe.  $\frac{\partial}{\partial t} \int \rho d\tau = \frac{\partial q}{\partial t}$  where  $q$  is the charge and  $\int \nabla \cdot \mathbf{j} d\tau = \oint \mathbf{j} \cdot d\mathbf{A}$  according to Gauss' divergence theorem. We are integrating over the whole universe and so  $\oint d\mathbf{A}$  covers the 'surface area' of the universe at infinity. But by the time we reach infinity all local currents will have died off to zero and so  $\oint \mathbf{j} \cdot d\mathbf{A} = 0$  yielding

$$\frac{\partial q}{\partial t} = 0 \quad (3.28)$$

which is the global conservation of charge law. It says that the total charge of the universe is constant.

Finally, let's go back and look at Ampère's law. The original form of Ampère's law didn't have the second term. It actually read  $\nabla \times \mathbf{B} = \frac{4\pi k}{gc^2} \mathbf{j}$ . From our above discussion this would have lead to  $\nabla \cdot \mathbf{j} = 0$ . Thus the original form of Ampère's law violated charge conservation [Guidry p.74]. Maxwell added the term  $-\frac{\partial \mathbf{E}}{\partial t}$ , which is now called Maxwell's displacement current. Writing  $\mathbf{j}_D \equiv \frac{1}{4\pi k} \frac{\partial \mathbf{E}}{\partial t}$ , Ampère's law is  $\nabla \times \mathbf{B} = \frac{4\pi k}{gc^2} (\mathbf{j} + \mathbf{j}_D)$ , or in integral form  $\oint \mathbf{B} \cdot d\mathbf{l} = \frac{4\pi k}{gc^2} (i + i_D)$  Maxwell's addition of the displacement current made Ampère's law agree with conservation of charge.

### 3.4 Electromagnetic Waves

Just as conservation of charge is an immediate consequence of Maxwell's equations, so too is the existence of electromagnetic waves, the immediate interpretation of light as an electromagnetic wave. This elucidation of the true nature of light is one of the great triumphs of classical electrodynamics as embodied in Maxwell's equations.

First let us recall that the 1-dimensional wave equation is

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (3.29)$$

where  $\psi(x, t)$  represents the wave and  $v$  is the speed of the wave. Contrast this to several other well known equations such as the heat equation [5]

$$\frac{\partial^2 \psi}{\partial x^2} - \frac{1}{v} \frac{\partial \psi}{\partial t} = 0 \quad (3.30)$$

or the Schrödinger equation [17]

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} + V\psi = i\hbar \frac{\partial \psi}{\partial t} \quad (3.31)$$

or the Klein-Gordon equation [17]

$$(\square^2 + m^2)\phi = 0 \quad (3.32)$$

where

$$\square^2 \equiv \frac{1}{v^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \quad (3.33)$$

In 3-dimensions the wave equation is

$$\nabla^2 \psi - \frac{1}{v^2} \frac{\partial^2 \psi}{\partial t^2} = 0 \quad (3.34)$$

We would like to think about light travelling in the vacuum of free space far away from sources of charge or current (which actually do produce the electromagnetic waves in the first place). Thus we set  $\rho = j = 0$  in Maxwell's equations, giving Maxwell's equations in free space as

$$\nabla \cdot \mathbf{E} = 0, \quad (3.35)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.36)$$

$$\nabla \times \mathbf{E} + g \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (3.37)$$

$$\nabla \times \mathbf{B} - \frac{1}{gc^2} \frac{\partial \mathbf{E}}{\partial t} = 0 \quad (3.38)$$

Taking the curl of Faraday's law gives  $\nabla \times (\nabla \times \mathbf{E}) + g \frac{\partial}{\partial t} (\nabla \times \mathbf{B}) = 0$  and substituting  $\nabla \times \mathbf{B}$  from Ampère's law gives  $\nabla \times (\nabla \times \mathbf{E}) + \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$ .

However  $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$  because of (3.35). (**do Problem 3.2**). Thus we have

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad (3.39)$$

and with the same analysis (**do Problem 3.3**)

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \quad (3.40)$$

showing that the electric and magnetic fields correspond to waves propagating in free space at speed  $c$ . Note that the constant  $g$  cancels out, so that these wave equations look the same in all units. It turns out that the permeability and permittivity of free space have the values such that  $1/\sqrt{\mu_0 \epsilon_0}$  equals the speed of light ! Thus the identification was immediately made that these electric and magnetic waves *are* light.

Probably the physical meaning of Maxwell's equations is unclear at this stage. Don't worry about this yet. The purpose of this chapter was to give a brief survey of Maxwell's equations and some immediate consequences. We shall study the physical meaning and solutions of Maxwell's equations in much more detail in the following chapters.

### 3.5 Scalar and Vector Potential

Remember our theorems in potential theory (Section 2.7) ? These were i)  $\mathbf{C} = -\nabla \phi$  iff  $\nabla \times \mathbf{C} = 0$  and ii)  $\mathbf{G} = \nabla \times \mathbf{F}$  iff  $\nabla \cdot \mathbf{G} = 0$ .

From the second theorem the magnetic Gauss law (3.2) immediately implies that  $\mathbf{B}$  can be written as the curl of some other vector  $\mathbf{A}$  as

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (3.41)$$

The vector  $\mathbf{A}$  is called the *magnetic vector potential* which we shall use often in the following chapters. Notice that (3.41) is the same in all units.

Looking at Maxwell's equations we don't see any curls equal to zero and so it looks like we won't be using the first theorem. But wait. We have just seen that  $\mathbf{B} = \nabla \times \mathbf{A}$ , so let's put this into Faraday's law as  $\nabla \times \mathbf{E} + g \frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = 0$  or

$$\nabla \times \left( \mathbf{E} + g \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad (3.42)$$

Now we have a vector  $\mathbf{C} \equiv \mathbf{E} + g \frac{\partial \mathbf{A}}{\partial t}$  whose curl is zero and therefore  $\mathbf{C}$  can be written as the gradient of some scalar function (let's call it  $V$ ) as  $\mathbf{C} = -\nabla V$ , from which it follows that

$$\mathbf{E} = -\nabla V - g \frac{\partial \mathbf{A}}{\partial t} \quad (3.43)$$

The scalar  $V$  is called the *electric scalar potential*, and this equations *does* depend on the unit system via the appearance of the constant  $g$ .

In the next two chapters we will be studying *time-independent* problems, in which all time derivatives in Maxwell's equations are zero. The time-independent Maxwell's equations are

$$\nabla \cdot \mathbf{E} = 4\pi k \rho, \quad (3.44)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (3.45)$$

$$\nabla \times \mathbf{E} = 0, \quad (3.46)$$

and

$$\nabla \times \mathbf{B} = \frac{4\pi k}{gc^2} \mathbf{j}, \quad (3.47)$$

implying that

$$\mathbf{E} = -\nabla V \quad (3.48)$$

only. This equation is independent of units.

We shall be using the scalar and vector potentials often in the following chapters where their meaning will become much clearer.



## Chapter 4

# ELECTROSTATICS

### 4.1 Equations for electrostatics

Electrostatics is a study the electric part of Maxwell's equations when all time derivatives are set equal to zero. Thus the two equations for the electric field are Gauss' law

$$\nabla \cdot \mathbf{E} = 4\pi k\rho, \quad (4.1)$$

and

$$\nabla \times \mathbf{E} = 0 \quad (4.2)$$

from Faraday's law yielding the scalar potential as

$$\mathbf{E} \equiv -\nabla V \quad (4.3)$$

(or from setting  $\frac{\partial \mathbf{A}}{\partial t}$  in equation (3.43)). Substituting this last result into Gauss' law yields *Poisson's equation* for the scalar potential

$$\nabla^2 V = -4\pi k\rho \quad (4.4)$$

which, in source free regions ( $\rho = 0$ ) gives *Laplace's equation*

$$\nabla^2 V = 0 \quad (4.5)$$

In integral form, Gauss law is (equation (3.23))

$$\Phi'_E \equiv \oint \mathbf{E} \cdot d\mathbf{A} = 4\pi kq \quad (4.6)$$

and Faraday's law is

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 \quad (4.7)$$



which also follows by using equation (4.2) in Stoke's curl theorem. To obtain equation (4.3) in integral form we integrate over  $\int d\mathbf{l}$  and use the fundamental theorem of gradients  $\int_a^P (\nabla V) \cdot d\mathbf{l} = V(P) - V(a)$  to give

$$V(P) - V(a) = - \int_a^P \mathbf{E} \cdot d\mathbf{l} \quad (4.8)$$

Notice from (4.3) that if we add a *constant* to the potential ( $V \rightarrow V + \text{constant}$ ) then the value of the electric field doesn't change. (Adding this constant is called a gauge transformation. Maxwell's equations are invariant under gauge transformations.) Being able to add a constant means that I can define the zero of potential to be anywhere I like. Thus let's define  $V(a) \equiv 0$  so that

$$V(P) = - \int_{\theta}^P \mathbf{E} \cdot d\mathbf{l} \quad (4.9)$$

where  $P$  is the point where the potential is being evaluated and  $a \equiv \theta$  is now a special point where the potential is zero. Typically the point  $\theta$  is either at the origin or at infinity. Thus (4.9) is the integral version of (4.3).

Finally let us obtain the integral form of Poisson's equation as

$$V(P) = k \int \frac{\rho}{R} d\tau \quad (4.10)$$

or  $k \int \frac{\lambda}{R} dl$  for line charges or  $k \int \frac{\sigma}{R} dA$  for surface charges, where  $\lambda$  and  $\sigma$  are the charge per unit length and charge per unit area respectively. The distance  $\mathbf{R}$  is shown in Fig. 4.1 and is defined as

$$\mathbf{R} \equiv \mathbf{r} - \mathbf{r}' \quad (4.11)$$

where  $\mathbf{r}$  is the displacement from the origin to point  $P$  and  $\mathbf{r}'$  is the displacement from the origin to the charge distribution. Thus  $\mathbf{r}$  is the displacement from the charge to point  $P$ . Notice that we didn't yet *prove* equation (4.10) like we did with the other equations. A proper solution of Poisson's equation involves Green functions [8] which are beyond the scope of this book. However, we shall provide a proper justification for equation (4.10) in Section 4.xx.

It is very useful to collect and summarize our results for electrostatics. Our basic quantities are the charge density  $\rho$ , the electric field  $\mathbf{E}$  and the scalar potential  $V$ .

The relation between  $\mathbf{E}$  and  $\rho$  is

$$\boxed{\nabla \cdot \mathbf{E} = 4\pi k\rho}$$

$$\boxed{\nabla \cdot \mathbf{E} = 4\pi k\rho \Phi'_E \equiv \oint \mathbf{E} \cdot d\mathbf{A} = 4\pi kq.}$$

(4.12)

The relation between  $\mathbf{E}$  and  $V$  is

$$\boxed{\mathbf{E} \equiv -\nabla V}$$

$$\boxed{V(P) = -\int_{\theta}^P \mathbf{E} \cdot d\mathbf{l}.}$$

(4.13)

The relation between  $V$  and  $\rho$  is

$$\boxed{\nabla^2 V = -4\pi k\rho}$$

$$\boxed{V(P) = k \int \frac{\rho}{R} d\tau.}$$

(4.14)

These relations are also summarized in Fig. 2.35 of the book by Griffiths [2].

Having established our basic equations for electrostatics, now let's investigate their solutions and physical meanings.

## 4.2 Electric Field

*Force* is a quantity that involves two bodies (force between two charged particles, or force between Earth and Moon), whereas *field* is a quantity pertaining to one object only. Coulomb's law for the electric force between two point charges is  $\mathbf{F} = k\frac{q_1q_2}{r^2}\hat{r}$  where  $r$  is the distance between the charges. The *definition* of electric field gets rid of one of these charges by dividing it out as

$$\mathbf{E} \equiv \frac{\mathbf{F}}{q} \quad (4.15)$$

so that the electric field surrounding a single point charge is  $\mathbf{E} = k\frac{q}{r^2}\hat{r}$ .

In order to calculate electric fields it is much more convenient to use Gauss' law in integral form  $\Phi'_E \equiv \oint \mathbf{E} \cdot d\mathbf{A} = 4\pi kq$  rather than the differential version. The integral  $\oint d\mathbf{A}$  forms a closed surface, which is often called a Gaussian surface which fully encloses the charge  $q$ . The key trick in using Gauss' law is to *always choose a Gaussian surface so that  $\mathbf{E}$  and  $d\mathbf{A}$  are parallel or perpendicular to each other*, because then either  $\mathbf{E} \cdot d\mathbf{A} = EdA \cos 0^\circ = EdA$  or  $\mathbf{E} \cdot d\mathbf{A} = EdA \cos 90^\circ = 0$  and one won't have to worry about vectors. The way to ensure that  $\mathbf{E}$  and  $d\mathbf{A}$  are parallel or perpendicular is to *choose the Gaussian surface to have the same symmetry as the charge distribution*. We illustrate this in the following examples.

**Example 4.2.1** Prove Coulomb's law.

**Solution** Coulomb's law gives the force between two point charges. Thus we first need the electric field due to one point charge. We *know* that the field surrounding a point charge is as shown in Fig. 4.2. The field points radially outward. The only surface for which  $d\mathbf{A}$  points radially outward also is the sphere (also shown in the figure) which has the same symmetry as the point charge. If we had drawn a cube as our Gaussian surface (which would still give the correct answer, but only with a much more difficult calculation) then  $\mathbf{E}$  and  $d\mathbf{A}$  would not be parallel everywhere. (*Exercise:* draw such a picture to convince yourself.)

Also note that the Gaussian surface has been drawn to intersect the observation point  $P$ . We always do this, so that the distance to the observation point is the same as the size of the Gaussian surface. Now  $\Phi'_E \equiv \oint \mathbf{E} \cdot d\mathbf{A}$  is easy. For a sphere  $dA = r^2 \sin\theta d\theta d\phi$  from equation (2.75). Note that on this Gaussian

sphere the *magnitude of the electric field is constant and therefore may be taken outside the integral* to give  $\Phi'_E = E \int r^2 \sin \theta d\theta d\phi = E4\pi r^2$  (as shown at the end of Section 2.8.4). Thus

$$E = k \frac{q}{R^2} \quad (4.16)$$

or, from our picture

$$\mathbf{E} = k \frac{q}{R^2} \hat{\mathbf{R}} \quad (4.17)$$

and using the definition of electric field  $\mathbf{E} \equiv \frac{\mathbf{F}}{q}$  gives

$$\mathbf{F} = k \frac{q_1 q_2}{R^2} \hat{\mathbf{R}} \quad (4.18)$$

Thus we see how Gauss' law yields Coulomb's law.

**Example 4.2.2** Find the electric field due to a charged plate of infinite area.

**Solution** The electric field lines due to a positively charge infinite plate are shown in Fig 4.3 together with a Gaussian surface of the same symmetry, which is a cylinder, but a square box would have done equally well. The angle between  $d\mathbf{A}_3$  (the area vector of the rounded edge) and  $\mathbf{E}$  is  $90^\circ$  and therefore  $\mathbf{E} \cdot d\mathbf{A}_3 = 0$ . Both  $d\mathbf{A}_2$  and  $d\mathbf{A}_1$  are parallel to  $\mathbf{E}$  so that  $\mathbf{E} \cdot d\mathbf{A}_2 = E dA_2$  and  $\mathbf{E} \cdot d\mathbf{A}_1 = E dA_1$ . Furthermore, on the areas  $\mathbf{A}_1$  and  $\mathbf{A}_2$  the electric field is constant (independent of  $R$ ) so that  $\mathbf{E}$  may be taken outside the integral to give  $\Phi'_E \equiv \oint \mathbf{E} \cdot d\mathbf{A} = \int \mathbf{E} \cdot d\mathbf{A}_1 + \int \mathbf{E} \cdot d\mathbf{A}_2 + \int \mathbf{E} \cdot d\mathbf{A}_3 = E \int dA_1 + E \int dA_2 + 0 = 2EA$  where  $A$  is the area of the circular ends. Thus  $2EA = 4\pi kq$ . The same area  $A$  intersects the charged plate so that we can define the surface charge density  $\sigma \equiv \frac{q}{A}$  as the charge per unit area giving

$$E = 2\pi k\sigma \quad (4.19)$$

or

$$\mathbf{E} = 2\pi k\sigma \hat{\mathbf{R}} \quad (4.20)$$

Note that here the electric field is independent of the distance from the plate.

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*Exercise:* A) Explain why the electric field is independent of the distance from the plate. B) Infinite plates don't exist. Why is our problem still of practical importance? C) A capacitor consists of a positive plate and a negative plate separated by a distance  $d$ . What is the electric field inside and outside the capacitor?

(do Problems 4.1, 4.2, 4.3)

### 4.3 Electric Scalar potential

In classical mechanics one always has the option of solving problems with either force methods or energy methods. The force methods involve the use of Newton's law  $\mathbf{F} = m\mathbf{a}$ , while the energy methods use conservation of energy. (The sophisticated version of energy methods is expressed in the re-formulations of mechanics via Lagrangian and Hamiltonian dynamics).

In electrodynamics 'force' methods involve calculation of electric field (trivially related to force via  $\mathbf{E} \equiv \frac{\mathbf{F}}{q}$ ). Alternatively the 'energy methods' are based upon calculation of the *electric potential*  $V$ . If we have the electric potential it is then easy to get the field just from  $\mathbf{E} = -\nabla V$ .

A word on naming things. The force between two charged particles is  $\mathbf{F} = k\frac{q_1q_2}{r^2}\hat{r}$  and the potential *energy* for a conservative force is always  $\mathbf{F} = -\nabla U$  giving  $U = k\frac{q_1q_2}{r}$  as the potential energy between two point charges. We defined the electric field intrinsic to a single charge as  $\mathbf{E} \equiv \frac{\mathbf{F}}{q}$  and similarly we can define (actually it follows) the potential  $V \equiv \frac{U}{q}$  which is sort of like the 'energy' intrinsic to a single charge. (Note that in MKS units, the potential has units of volts, so the  $V$  symbol is a good one. Actually a better name for  $V$  is voltage rather than potential, because then we don't get mixed up with potential energy.)

In mechanics it is very often useful to have both the concept of force and the concept of energy. Similarly in electrodynamics it turns out to be very useful to have both the concept of field and the concept of potential. We shall illustrate the use of potential in the following example.

**Example 4.3.1** Calculate the electric potential inside and outside a uniformly charged spherical shell of radius  $a$ . A) Use equation (4.13). B) Use equation (4.14). (See examples 6 and 7 in chapter 2 of the book by Griffiths [2].)

**Solution** A) In problem 4.1 the electric field was calculated as

$\mathbf{E}(r < a) = 0$  inside the sphere and  $\mathbf{E}(r > a) = k\frac{q}{r^2}\hat{r} \equiv k\frac{q}{r^2}\hat{e}_r$  outside. Let us use equation (4.13) in the form

$$V(P) = -\int_{\infty}^P \mathbf{E} \cdot d\mathbf{l} \quad (4.21)$$

where we have chosen the reference point of zero potential at infinity, i.e.  $V(\infty) = 0$ .

In spherical polar coordinates  $d\mathbf{l} = dl_r\hat{e}_r + dl_\theta\hat{e}_\theta + dl_\phi\hat{e}_\phi$ , so that for points outside the sphere  $\mathbf{E} \cdot d\mathbf{l} = k\frac{q}{r^2}\hat{e}_r \cdot (dr\hat{e}_r + dl_\theta\hat{e}_\theta + dl_\phi\hat{e}_\phi) = k\frac{q}{r^2}dr$  giving

$$-kq \int_{\infty}^R \frac{1}{r^2} dr = \frac{kq}{R} \quad (4.22)$$

*This is also the potential of a point charge.*

Notice that we have integrated all the way out from  $\infty$  up to the point  $P$  at  $R$ , but we didn't cross the surface of the sphere in our journey because the point  $P$  at  $R$  is outside the sphere.

Continuing our journey across the spherical boundary and inside the sphere gives us the potential inside. But now  $\mathbf{E}$  has two different values depending upon whether we are inside or outside so that  $V(r < a) = -[\int_{\infty}^a \mathbf{E} \cdot d\mathbf{l} + \int_a^r \mathbf{E} \cdot d\mathbf{l}]$  but the second integral is zero because  $\mathbf{E}(r < a) = 0$ . The first integral is just the one we did above except the upper limit of integration is  $a$  rather than  $R$ . Thus  $V(r < a) = \frac{kq}{a}$  showing that the potential is *not* zero inside the sphere but rather is a constant. Using  $\mathbf{E} = -\nabla V$  does give zero electric field inside the sphere.

B) This part of the problem shows how to calculate the potential if we haven't already got the electric field  $\mathbf{E}$ . The solution to Poisson's equation is given in equation (4.14) as  $V(P) = k \int \frac{\rho}{R} d\tau$ . It is important to note that  $\mathbf{R}$  is the vector from a point within the charge distribution to the observation point  $P$ , whereas  $d\tau$  is an integral over the charge density  $\rho$  only. (See Fig. 4.1.) *An easy understanding of equation (4.14) is to consider it simply as a generalization of equation (4.22).* For a surface charge density  $\sigma$  (charge per unit area),  $\rho d\tau$  is simply replaced with  $\sigma dA$ .

To make things easy for us let's put the point  $P$  on the  $z$  axis and the origin at the center of the sphere as shown in Fig. 4.4. Using the law of cosines we have  $R^2 = z^2 + a^2 - 2za \cos \theta$ . In spherical

polar coordinates the appropriate area element is  $dA = dl_\theta dl_\phi = a^2 \sin\theta d\theta d\phi$  to give

$$V(z) = k\sigma \int \frac{a^2 \sin\theta d\theta d\phi}{\sqrt{z^2 + a^2 - 2za \cos\theta}} = 2\pi a^2 k\sigma \int_0^\pi \frac{\sin\theta d\theta}{\sqrt{z^2 + a^2 - 2za \cos\theta}}.$$

The integral is easy to do, giving

$$V(z) = \frac{2\pi ak\sigma}{z} [\sqrt{(z+a)^2} - \sqrt{(z-a)^2}] = \frac{2\pi ak\sigma}{z} [\sqrt{(z+a)^2} - \sqrt{(a-z)^2}].$$

For points outside the sphere ( $z > a$ ) we use the first expression (taking only the positive roots) to give

$$V(z > a) = \frac{2\pi ak\sigma}{z} [(z+a) - (z-a)] = \frac{4\pi a^2 k\sigma}{z} = \frac{kq}{z} \text{ (where we have used } \sigma = \frac{q}{4\pi a^2} \text{ and inside } V(z < a) = \frac{2\pi ak\sigma}{z} [(z+a) - (a-z)] = 4\pi ak\sigma = \frac{kq}{a}.$$

Of course the orientation of our axes is arbitrary so that we may replace  $V(z > a) = \frac{kq}{z}$  by  $V(R > a) = \frac{kq}{R}$  and  $V(z < a) = \frac{kq}{a}$  by  $V(R < a) = \frac{kq}{a}$  in agreement with our results above. (**do Problems 4.4 and 4.5**).

## 4.4 Potential Energy

Work is defined as

$$W \equiv \int \mathbf{F} \cdot d\mathbf{l} \quad (4.23)$$

and substituting the right hand side of  $\mathbf{F} = m\mathbf{a}$  the integral (let's just do 1 dimension) becomes  $m \int a dx = m \int \frac{dv}{dt} dx = m \int \frac{dv}{dx} \frac{dx}{dt} dx = m \int v \frac{dv}{dx} dx = m \int_i^f v dv = \Delta T$  where the kinetic energy is defined as  $T \equiv \frac{1}{2}mv^2$  and  $\Delta T \equiv T_f - T_i$ . Thus work is *always* equal to the change in kinetic energy. Now let's break the work up into a conservative ( $C$ ) and a non-conservative ( $NC$ ) piece as

$$\begin{aligned} W &\equiv W_C + W_{NC} = \int \mathbf{F} \cdot d\mathbf{l} = \Delta T \\ &= \int \mathbf{F}_C \cdot d\mathbf{l} + W_{NC} \end{aligned} \quad (4.24)$$

and let's define the conservative piece of the force ( $\mathbf{F} = \mathbf{F}_C + \mathbf{F}_{NC}$ ) as

$$\mathbf{F}_C \equiv -\nabla U \quad (4.25)$$

where  $U$  is the potential *energy* (as opposed to the potential or voltage). This relation between conservative force and potential energy is analogous to the relation between electric field and potential (voltage),  $\mathbf{E} = -\nabla V$ . From potential theory it immediately follows that

$$\begin{aligned} \nabla \times \mathbf{F}_C &= 0, \\ \Rightarrow \oint \mathbf{F}_C \cdot d\mathbf{l} &= 0. \end{aligned} \quad (4.26)$$

(In electromagnetic theory we *started* with  $\nabla \times \mathbf{E} = 0$  and then got  $\mathbf{E} = -\nabla V$ . Above we are starting with a definition  $\mathbf{F}_C \equiv -\nabla U$  and then get  $\nabla \times \mathbf{F}_C = 0$ .) The conservative part of the work becomes

$$W_C = \int \mathbf{F}_C \cdot d\mathbf{l} \equiv -\int_i^f \nabla U \cdot d\mathbf{l} = -(U_f - U_i). \quad (4.27)$$

Thus

$$W_C = -\Delta U \quad (4.28)$$

from the fundamental theorem of gradients. Substituting back into (4.24) we have  $-\Delta U + W_{NC} = \Delta T$  or

$$\boxed{\Delta U + \Delta T = W_{NC}} \quad (4.29)$$

which is the famous work-energy theorem.  $W_{NC}$  stands for non-conservative work such as heat, sound energy etc. If no heat or sound is involved then the *mechanical* energy ( $T + V$ ) is conserved via  $\Delta U + \Delta T = 0$ .

Equation (4.29) is also the First Law of Thermodynamics. To see this write  $W_{NC} = \Delta Q$  for the change in heat and  $W = \Delta T$  (remember it's always this). Then  $\Delta U + \Delta W = \Delta Q$  which is more often written as

$$dU + dW = dQ \quad (4.30)$$

We already have a formula for the kinetic energy as  $T = \frac{1}{2}mv^2$  which is *always* true. What about a formula for the potential energy  $U$ ? Well our formula for  $U$  will depend on the force. Recall that the formula for  $T$  came from integrating the right hand side of  $\mathbf{F} = m\mathbf{a}$  and  $U$  comes from the left hand side. That is why  $T$  is always the same,  $\frac{1}{2}mv^2$ , but  $U$  changes depending on what  $\mathbf{F}$  is.



In what we study below we are interested in how much work  $I$  must do *against* a force. Thus

$$\mathbf{F} \cdot d\mathbf{l} = F dl \cos \theta = -F dl \quad (4.31)$$

because  $\theta = 180^\circ$  if I am pulling in the direction  $d\mathbf{l}$  but the force is pulling in the opposite direction. Having decided upon the signs, then  $\mathbf{F}$  in the above equation is now a *scalar*. For example, for a spring we often write  $\mathbf{F} = -k\mathbf{x}$ , but since I have already dealt with signs I use  $F = kx$  in equation (4.31).

For all the cases below the conservative work is

$$W_C = -\Delta U = \int_i^f \mathbf{F} \cdot d\mathbf{l} = -\int_i^f F dl \quad (4.32)$$

giving

$$\Delta U = \int_i^f F dl \quad (4.33)$$

*Gravity near the surface of Earth* ( $F = mg$ ). Here we have

$$\begin{aligned} U_f - U_i &= mg \int_i^f dl \\ &= mgl_f - mgl_i \end{aligned} \quad (4.34)$$

and leaving off the  $i$  and  $f$  symbols yields

$$U = mgl \quad (4.35)$$

in general, which is just the formula for the potential energy that we learnt in freshman physics (potential energy =  $mgh$ ). Note that the zero of potential energy is at  $l = 0$ , i.e.  $U(l = 0) = 0$ .

*Simple Harmonic Oscillator*. ( $F = kx$ ).

$$\begin{aligned} U_f - U_i &= k \int_i^f x dx \\ &= \frac{1}{2} kx_f^2 - \frac{1}{2} kx_i^2 \end{aligned} \quad (4.36)$$

yielding

$$U = \frac{1}{2} kx^2 \quad (4.37)$$

where the zero of potential energy is  $U(x = 0) = 0$ .

*Universal Gravitation.* ( $F = G\frac{m_1m_2}{r^2}$ ).

$$\begin{aligned} U_f - U_i &= Gm_1m_2 \int_i^f \frac{dr}{r^2} \\ &= -G\frac{m_1m_2}{r_f} + G\frac{m_1m_2}{r_i} \end{aligned} \quad (4.38)$$

yielding

$$U = -G\frac{m_1m_2}{r} \quad (4.39)$$

where now the zero point of potential energy is at infinity, namely  $U(r = \infty) = 0$ .

*Coulomb's Law.* Here we have a problem. The sign of the force is different depending on whether we have like or unlike charges, giving a repulsive or attractive force respectively. The easiest way not to get confused is to copy gravity which is always attractive. Thus the formulas for unlike charges will look just like gravity and for like charges we just change whatever sign shows up for gravity. Thus 'copying' equation (4.39)

$$U = -k\frac{q_1q_2}{r} \quad (4.40)$$

for unlike charges (attractive force). Therefore we must have

$$U = +k\frac{q_1q_2}{r} \quad (4.41)$$

for like charges.

*All Forces.* It's a bit of a nuisance having to always go through the effort of the above calculations for potential energy. There is a good memory device for getting the answer instantly. The memory device is

$$F = |\mathbf{F}| = Cx^n \quad \Rightarrow \quad U = C\frac{x^{n+1}}{n+1} \quad (4.42)$$

which makes 'sense' from  $\mathbf{F} = -\nabla U$ . In using this memory device  $|\mathbf{F}|$  means that we ignore any sign for the force.  $x$  is the corresponding distance.

*Gravity near Earth.* ( $F = |\mathbf{F}| = mg$ ). Thus  $n = 0$  because a distance does not appear in the force. Here  $C = mg$  giving  $U = mg\frac{l^1}{1} = mgl$  in agreement with (4.34).

*Simple Harmonic Oscillator.* ( $F = |\mathbf{F}| = kx$ ). We have  $n = 1$  and  $C = k$  giving  $U = k\frac{x^{1+1}}{2} = \frac{1}{2}kx^2$  in agreement with (4.36).

*Universal Gravitation.* ( $F = |\mathbf{F}| = G\frac{m_1m_2}{r^2}$ ). We have  $n = -2$  and  $C = Gm_1m_2$  giving  $U = Gm_1m_2\frac{r^{-2+1}}{-2+1} = -G\frac{m_1m_2}{r}$  in agreement with (4.38).

#### 4.4.1 Arbitrariness of zero point of potential energy

The zero point of potential energy and potential (point at which they are equal to zero) is supposed to be completely arbitrary [6] Pg. 370, yet from the above analysis it looks like the zero point was automatically determined. (We are working with  $\mathbf{F} = -\nabla U$  but *identical* results hold for  $\mathbf{E} = -\nabla V$ . Remember  $\mathbf{E} \equiv \frac{\mathbf{F}}{q}$  and  $V \equiv \frac{U}{q}$ .)

We have

$$\begin{aligned}\mathbf{F} &= -\nabla U \\ \Rightarrow -\Delta U &= \int \mathbf{F} \cdot d\mathbf{l} \\ U_i - U_f &= \int_i^f \mathbf{F} \cdot d\mathbf{l}\end{aligned}\tag{4.43}$$

Let's choose  $U_i \equiv 0$  at the special point  $i_0$  giving

$$U_f = -\int_{i_0}^f \mathbf{F} \cdot d\mathbf{l}\tag{4.44}$$

with  $U(i_0) \equiv 0$ , or

$$U(P) = -\int_{i_0}^P \mathbf{F} \cdot d\mathbf{l}\tag{4.45}$$

analogous to equations (4.13) and (4.21). With  $\mathbf{F} \cdot d\mathbf{l} = F dl \cos \theta = -F dl$  as we chose above we would have

$$U(P) = \int_{i_0}^P F dl\tag{4.46}$$

in perfect agreement with our results before. In fact *this formula shows why our memory device works*. If the point  $i_0 = \infty$  then

$$U(P) = -\int_{\infty}^P \mathbf{F} \cdot d\mathbf{l}\tag{4.47}$$

with  $U(\infty) \equiv 0$  the same as (4.21).

#### 4.4.2 Work done in assembling a system of charges

*The work done in assembling a system of charges is identical to the energy stored in a system of charges.* The potential energy stored in a system of

two like charges, from equation (4.41) is  $U = +k\frac{q_1q_2}{r}$ . The work required to assemble these two like charges is just  $W = U = k\frac{q_1q_2}{r}$ . But  $V \equiv \frac{U}{q}$  and so

$$W = qV \quad (4.48)$$

is the work required to put a single charge into potential  $V$ . The work required to assemble a *system* of charges is

$$W = \frac{1}{2} \sum_i q_i V_i = \frac{1}{2} \int \rho V d\tau \quad (4.49)$$

where the  $\frac{1}{2}$  factor comes from double counting [2], Pg. 93-94. We re-write this using  $\nabla \cdot \mathbf{E} = 4\pi k\rho$  to give  $W = \frac{1}{2} \frac{1}{4\pi k} \int (\nabla \cdot \mathbf{E}) V d\tau$ . Now

$$\nabla \cdot (\mathbf{E}V) = (\nabla \cdot \mathbf{E})V + \mathbf{E} \cdot (\nabla V) = (\nabla \cdot \mathbf{E})V - E^2$$

(do Problem 4.6) so that

$$W = \frac{1}{8\pi k} [\int \nabla \cdot (\mathbf{E}V) d\tau + \int E^2 d\tau] = \frac{1}{8\pi k} [\int V \mathbf{E} \cdot d\mathbf{A} + \int E^2 d\tau].$$

However the surface integral is zero [2], Pg. 95 giving

$$W = \frac{1}{8\pi k} \int_{all\ space} E^2 d\tau \quad (4.50)$$

showing that the energy *density*  $\mathcal{E}$  of the electric field varies as

$$\mathcal{E} \propto E^2.$$

**Example 4.3.2** Calculate the energy stored in the charged spherical shell discussed in Example 4.3.1 using both (4.49) and (4.50). (See example 8 in chapter 2 of the book by Griffiths [2].)

**Solution** A)  $W = \frac{1}{2} \int \rho V d\tau = \frac{1}{2} \int \sigma V dA$  for a surface charge. The energy we are calculating is actually the work necessary to bring a system of charges together to *make* the spherical shell in the first place. Thus we want the potential  $V$  at the surface of the shell. This is  $V = \frac{kq}{a}$  which is a constant. Therefore  $W = \frac{1}{2} \frac{kq}{a} \int \sigma dA = k\frac{q^2}{2a}$ .

B)  $W = \frac{1}{8\pi k} \int_{all\ space} E^2 d\tau$  where the integration extends over all space. Inside the sphere  $\mathbf{E} = 0$  and outside  $\mathbf{E} = k\frac{q}{r^2} \hat{e}_r$  giving  $E^2 = k^2 \frac{q^2}{r^4}$ . Thus  $W = \frac{kq^2}{8\pi} \int \frac{r^2 \sin\theta dr d\theta d\phi}{r^4} = 2kq^2 \int_a^\infty \frac{dr}{r^2} = k\frac{q^2}{2a}$  in agreement with our result above. (do Problem 4.7).

## 4.5 Multipole Expansion

Griffiths Chpt. 3 (leave out this time).

## Chapter 5

# Magnetostatics

We shall develop the theory in this chapter in exact analogy to our study of electrostatics.

### 5.1 Equation for Magnetostatics

Magnetostatics is a study of the magnetic part of Maxwell's equation when all time derivatives are not equal to zero. Thus two equations for the magnetic field are Gauss' Law

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

(5.1)

and Ampère's Law

$$\boxed{\nabla \times \mathbf{B} = 4\pi k_m \mathbf{j}}$$

(5.2)

where

$$k_m \equiv \frac{k}{gc^2}. \quad (5.3)$$

### 5.1.1 Equations from Ampère's Law

The *solution to Ampère's Law is just the Biot-Savart Law*

$$\boxed{\mathbf{B} = k_m \int \frac{\mathbf{j} \times \hat{R}}{R^2} d\tau'}$$
(5.4)

which is checked by taking  $\nabla \times \mathbf{B}$  and showing that the right hand side of (5.2) is obtained. (do **problem 5.1**)

Ampère's Law in integral form is

$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = 4\pi k_m i}$$
(5.5)

The Biot-Savart law can always be saved to obtain the magnetic field, but the calculations may often be difficult. If a high degree of symmetry exists it is much *easier* to calculate the magnetic field using Ampère's law (in integral form).

### 5.1.2 Equations from Gauss' Law

Gauss' law in integral form is

$$\oint \mathbf{B} \cdot d\mathbf{a} = 0$$
(5.6)

but this isn't very useful for us. Much more useful is the vector potential

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}}$$
(5.7)

which is an immediate consequence of  $\nabla \cdot \mathbf{B}$ . Substituting into Ampère's law yields Poisson's equation

$$\boxed{\nabla^2 \mathbf{A} = -4\pi k_m \mathbf{j}}$$

(5.8)

where we have assumed *Coulomb gauge* [?] (pg.227) in which

$$\nabla \cdot \mathbf{A} = 0 \quad (5.9)$$

For  $\mathbf{j} = 0$ , Poisson's equation becomes Laplace's equation  $\nabla^2 \mathbf{A} = 0$ . The solution to  $\mathbf{B} = \nabla \times \mathbf{A}$  is obviously

$$\boxed{\mathbf{A} = \frac{1}{4\pi} \int \frac{\mathbf{B} \times \hat{R}}{R^2} d\tau'} \quad (5.10)$$

The reason this is obvious from comes looking at the solution (5.4) to Ampère's law.

The solution to Poisson's equation is obviously

$$\mathbf{A} = k_m \int \frac{\mathbf{j} d\tau'}{R} \quad (5.11)$$

The reason this is obvious comes from looking at the solution (??) to Poisson's equation in electrostatics.

We have now established our basic equation for magnetostatics. now let's investigate their solutions and physical meaning.

## 5.2 Magnetic Field from the Biot-Savart Law

We shall illustrate the use of the Biot-Savart law by calculating the magnetic field due to several current distributions.

Before proceeding however recall that in electrostatics we need  $q = \int \rho d\tau$  or  $\int \sigma da$  or  $\int \lambda dl$  for volume, surface and line charges respectively where  $\rho$  is the charge per unit volume,  $\sigma$  is charge per unit area and  $\lambda$  is charge per unit length. In the Biot-Savart law we have  $\int \mathbf{j} d\tau$  where  $\mathbf{j}$  is the current per unit area, which can be written as

$$\mathbf{j} = \rho \mathbf{v} \quad (5.12)$$

where  $\rho$  is charge per volume and  $\mathbf{v}$  is the speed of the current. The units of  $\mathbf{j} = \rho \mathbf{v}$  are  $\frac{\text{Coulomb}}{m^3} = \frac{\text{Coulomb}}{\text{sec}} \frac{1}{m^2}$  which is current per area. Thus  $\int \mathbf{j} d\tau$  will



have units of Amp-m which is current times length or *charge times speed*, i.e.

$$q\mathbf{v} \sim \int \mathbf{j}d\tau \quad (5.13)$$

Consequently for surfaces and lines we have  $q\mathbf{v} \sim \int \mathbf{j}d\tau \sim \int \mathbf{k}da \sim \int \mathbf{i}dl$  where  $\mathbf{j}$ =current/area,  $\mathbf{k}$ =current/length and  $\mathbf{i}$ =current only.

**Example 5.2.1** Calculate the magnetic field due to a steady current in a long, thin wire.

**solution** A thin wire will carry a current  $\mathbf{i}$  so that the Biot-Savart law is

$$\begin{aligned} \mathbf{B} &= k_m \int \frac{\mathbf{i} \times \hat{R}}{R^2} dl \\ &= k_m i \int \frac{dl \times \hat{R}}{R^2} \end{aligned}$$

From Fig. 5.1 we see that  $dl \times \hat{R}$  points out of the page and has magnitude  $|dl \times \hat{R}| = dl \sin \phi$ . For an infinitely long wire we will want to integrate  $\int_{-\infty}^{\infty} dl$  but this is more clearly accomplished by  $\int_{-\pi/2}^{\pi/2} d\theta$ . Thus let's use  $\theta$  as the angle variable instead of  $\phi$ . Thus  $|dl \times \hat{R}| = dl \sin \phi = dl \sin(\pi/2 - \theta) = dl \cos \theta$ . Thus the magnitude of the magnetic field is  $B = k_m i \int \frac{dl \cos \theta}{R^2}$ , but  $\ell$  and  $\theta$  are not independent and so we express one in terms of the other so that  $dl \cos \theta = \frac{a}{\cos \theta} d\theta$ . However  $R = \frac{a}{\cos \theta}$  so that  $\frac{dl \cos \theta}{R^2} = \frac{1}{a} \cos \theta d\theta$  yielding  $B = \frac{k_m i}{a} \int_{\theta_1}^{\theta_2} \cos \theta d\theta$ , giving our final result

$$B = \frac{k_m i}{a} (\sin \theta_2 - \sin \theta_1) \quad (5.14)$$

for a wire of finite length. If the wire is infinitely long then  $\theta_2 = \pi/2$  and  $\theta_1 = -\pi/2$  so that

$$B = \frac{2k_m i}{a}. \quad (5.15)$$

To obtain the direction of  $\mathbf{B}$ , the right hand rule readily yields that  $\mathbf{B}$  consists of circular loops around the wire, as shown in Fig.5.2.

### 5.3 Magnetic Field from Ampère's Law

Ampère's law works similarly to Gauss' law for electrostatics. Instead of drawing a Gaussian surface we draw an Ampèrian loop with the same symmetry as the current distribution.

**Example 5.3.1** Work out the previous example but using Ampère's law for an infinitely long wire.

**solution** An Ampèrian loop drawn so that the angle between  $\mathbf{B}$  and  $d\ell$  is 0 is obviously a loop just like the magnetic loops of Fig.5.2. Thus  $\oint \mathbf{B} \cdot d\ell = B2\pi a = 4\pi k_m i$  giving  $B = \frac{2k_m i}{a}$  in agreement with (5.15). We see that Ampère's law gives us the result *much* more quickly.

### 5.4 Magnetic Field from Vector Potential

We can also obtain the magnetic field if we have already calculated the vector potential from  $\mathbf{B} = \nabla \times \mathbf{A}$ , where  $\mathbf{A}$  is calculated first using (5.11).

**Example 5.4.1** A) Calculate the vector potential for the example discussed in Example 5.2.1. B) Calculate the magnetic field from  $\mathbf{B} = \nabla \times \mathbf{A}$  and show that it agrees with the previous result in Example 5.2.1. (see problem 5.25 of Griffiths)

**solution** See solutions in Griffiths

### 5.5 Units

Recall how our previous constants  $k$  and  $g$  came about.  $k$  was the arbitrary originally obtained in Coulomb's law  $F = \frac{q_1 q_2}{r^2}$ , appearing equivalently in  $\mathbf{E} = k \int \hat{R} \frac{\rho d\tau'}{R^2}$  or equivalently in Gauss' law  $\nabla \cdot \mathbf{E} = 4\pi k \rho$ . The constant  $g$  appeared in relating electric to magnetic phenomena via  $\nabla \times \mathbf{E} + g \frac{\partial \mathbf{B}}{\partial t} = 0$  and it serves to relate  $\mathbf{E}$  to  $\mathbf{B}$ .

Just as Coulomb's law is fundamental to electrostatics, so too is the Biot-Savart law to magnetostatics. One might wonder therefore why a separate fundamental constant did not appear in the Biot-Savart law. Actually it did!

Equation (5.4) has the constant  $k_m$  appearing in it as  $\mathbf{B} = k_m \int \frac{\mathbf{j} \times \hat{\mathbf{R}}}{R^2} d\tau'$ . Thus just as  $k$  is the constant fundamental to Coulomb's law, so too  $k_m$  is the constant fundamental to the Biot-Savart law. However electric and magnetic phenomena are related and the constant  $g$  relates them. Obviously then we have three constants but we would expect only two of them to be truly independent. This is exemplified by our relation  $k_m \equiv \frac{k}{gc^2}$  in equation (5.3). Thus we wrote Maxwell's equations with  $k$  and  $g$ , but we could instead have used  $k_m$  and  $k$  or alternatively  $k_m$  and  $g$ . Thus we can expand Table 3.1 into Table 5.1.

	Heaviside-Lorentz	Gaussian (CGS)	SI (MKS)
$k$	$\frac{1}{4\pi}$	1	$\frac{1}{4\pi\epsilon_o}$
$g$	$\frac{1}{c}$	$\frac{1}{c}$	1
$k_m \equiv \frac{k}{gc^2}$	$\frac{1}{4\pi c}$	$\frac{1}{c}$	$\frac{1}{4\pi\epsilon_o c^2} = \frac{\mu_o}{4\pi}$

Table 5.1.

Note that in SI units  $k_m = \frac{\mu_o}{4\pi}$ , where  $\mu_o$  is the magnetic permeability of vacuum.

## Chapter 6

# ELECTRO- AND MAGNETOSTATICS IN MATTER

### 6.1 Units

In this chapter we shall be introducing action fields called *displacement field*  $\mathbf{D}$ , *polarization field*,  $\mathbf{P}$ , *magnetization*  $\mathbf{M}$  and another field  $\mathbf{H}$  which we shall simply call the  $\mathbf{H}$  field. They are related to electric and magnetic fields by the following definitions [10]

$$\mathbf{D} \equiv \epsilon_o \mathbf{E} + \lambda \mathbf{P} \quad (6.1)$$

and

$$\mathbf{H} = \frac{1}{\mu_o} \mathbf{B} - \lambda' \mathbf{M}, \quad (6.2)$$

where  $\epsilon_o$ ,  $\mu_o$ ,  $\lambda$  and  $\lambda'$  are proportionality constants [10].  $\mathbf{D}$  and  $\mathbf{P}$  are always shown to have the same units, as are  $\mathbf{B}$  and  $\mathbf{M}$ . The only difference between them are the *numerical* factors  $\lambda$  and  $\lambda'$ . In *rationalized* units (such as Heaviside-Lorentz and SI)  $\lambda \equiv \lambda' \equiv 1$ , whereas in *unrationalized* units (such as Gaussian)  $\lambda \equiv \lambda' \equiv 4\pi$ .

However  $\mathbf{D}$  and  $\mathbf{E}$  need not have the same units, nor need  $\mathbf{B}$  and  $\mathbf{H}$ . (However in Heaviside-Lorentz and Gaussian system they do have the same units.) The values of all the constants are listed in Table 6.1.

In SI units we usually just leave the symbols  $\epsilon_o$  and  $\mu_o$  instead of writing their values which are  $\epsilon_o = \frac{10^7}{4\pi(c)^2}$  and  $\mu_o = 4\pi \times 10^{-7}$  [10].

	Heaviside-Lorentz	Gaussian (CGS)	SI (MKS)
$k$	$\frac{1}{4\pi}$	1	$\frac{1}{4\pi\epsilon_o}$
$g$	$\frac{1}{c}$	$\frac{1}{c}$	1
$k_m \equiv \frac{k}{gc^2}$	$\frac{1}{4\pi c}$	$\frac{1}{c}$	$\frac{1}{4\pi\epsilon_o c^2} \equiv \frac{\mu_o}{4\pi}$
$\lambda$	1	$4\pi$	1
$\lambda'$	1	$4\pi$	1
$\epsilon_o$	1	1	$\epsilon_o$
$\mu_o$	1	1	$\mu_o$

Table 6.1: Table 6.1

For purposes of having our equations independent of units, we shall leave  $\epsilon_o$ ,  $\mu_o$ ,  $\lambda$  and  $\lambda'$  in all equations. Note that  $\epsilon_o$  is the permittivity of vacuum and  $\mu_o$  is the permeability of vacuum.

As we shall discuss below, the relations for a *linear* medium are *always* (for *any* unit system) [10]

$$\mathbf{D} = \epsilon \mathbf{E} \quad (6.3)$$

and

$$\mathbf{B} = \mu \mathbf{H} \quad (6.4)$$

and the constants  $\epsilon_o$ ,  $\mu_o$  are the vacuum values of  $\epsilon$ ,  $\mu$  and according to Jackson [10] the ratio of  $\epsilon/\epsilon_o$  is often called the *relation permeability* or just the *permeability*.

This whole discussion in this section simply serves to introduce units. We shall now discuss the *concepts* in much more detail.

## 6.2 Maxwell's Equations in Matter

### 6.2.1 Electrostatics

Suppose a dielectric material is placed in an external electric field as shown in Figure 6.1. Because the atoms and molecules within the solid contain positive and negative charges they will become slightly polarized due to the external field. This will lead to an overall polarization of the material with polarization vector  $\mathbf{P}$  which will be in a direction opposite to the external vector  $\mathbf{E}$  as shown in Figure 6.1. Thus the *net field*  $\mathbf{D}$  in the material will be different to  $\mathbf{E}$  and  $\mathbf{P}$ . This  $\mathbf{D}$  is called the displacement field.

Just as the strength of an electric field is due to a charge density via  $\nabla \cdot \mathbf{E} = 4\pi k\rho$ , so too will the fields  $\mathbf{P}$  and  $\mathbf{D}$  be due to *affective* charge densities. The polarization field will be due to the charges trapped (or *bound*) within the individual microscopic dipoles. If there still remains a residual  $\mathbf{D}$  in the material then this will be due to *non-bound* or *free* charges.

Thus the total charge density  $\rho$  is written in terms of free charge density  $\rho_f$  and bound charge density  $\rho_b$  as (true in *all* unit systems)

$$\rho \equiv \rho_f + \rho_b. \quad (6.5)$$

From our discussion above there are three fields in the dielectric. We have the external field  $\mathbf{P}$ , so that the *resultant* field in the medium is  $\mathbf{D}$  which will be a *linear combination* of  $\mathbf{E}$  and  $\mathbf{P}$ . Thus *define* (true for *all* units.)

$$\boxed{\mathbf{D} = \epsilon_o \mathbf{E} + \lambda \mathbf{P}} \quad (6.6)$$

where  $\epsilon_o$  and  $\lambda$  are, as yet, undetermined constants, *both* of whose values we are free to choose. The choices are listed in Table 6.1.

We would now like to find a Gauss' law for  $\mathbf{D}$ . Thus let's calculate

$$\nabla \cdot \mathbf{D} = \epsilon_o \nabla \cdot \mathbf{E} + \lambda \cdot \mathbf{P} = 4\pi k \epsilon_o \rho + \lambda \nabla \cdot \mathbf{P}$$

However in *all* unit systems (**do Problem 6.1**)

$$4\pi k \epsilon_o = \lambda \quad (6.7)$$

so that

$$\nabla \cdot \mathbf{D} = \lambda(\rho + \nabla \cdot \mathbf{P}) \quad (6.8)$$

$$= \lambda \rho_f + \lambda(\rho_b + \nabla \cdot \mathbf{P}). \quad (6.9)$$

Now we have  $\nabla \cdot \mathbf{D}$  related to  $\rho_f$  and  $\nabla \cdot \mathbf{P}$  is related to  $\rho_b$ . Evidently then require (*all units*)

$$\boxed{\nabla \cdot \mathbf{P} = -\rho_b}$$
(6.10)

where the minus sign can be understood because  $\mathbf{P}$  is in a direction opposite to  $\mathbf{E}$  (see Figure 6.1). Thus we finally obtain Gauss' law as (*all units*)

$$\boxed{\frac{1}{\epsilon_o} \nabla \cdot \mathbf{D} = 4\pi k \rho_f}$$
(6.11)

It probably makes sense that if we increase the external electric field then the polarization field would increase as well. A *linear* medium is one in which the polarization is directly proportional to the electric field and not, say, the square of the electric field. The constant of proportionality is the susceptibility. Thus for a *linear* medium (for *all units*)

$$\mathbf{P} \equiv \epsilon_o \chi_e \mathbf{E}$$
(6.12)

where  $\chi_e$  is called the *electrical susceptibility*. Thus in HL and CGS units ( $\epsilon_o = 1$ ) we would have  $\mathbf{P} = \chi_e \mathbf{E}$  and in MKS units we have  $\mathbf{P} = \epsilon_o \chi_e \mathbf{E}$ . Substituting (6.12) into (6.11) yields (for *all units*)

$$\mathbf{D} = \epsilon_o (1 + \lambda \chi_e) \mathbf{E} \equiv \epsilon \mathbf{E}$$
(6.13)

which serves to define the permittivity  $\epsilon$ . The *relative permittivity* or *dielectric constant* is defined as (*all units*)

$$\kappa \equiv \frac{\epsilon}{\epsilon_o} = 1 + \lambda \chi_e.$$
(6.14)

Thus in HL units we have  $\mathbf{D} = (1 + \chi_e) \mathbf{E}$  and  $\kappa = \epsilon = 1 + \chi$ . In CGS units  $\mathbf{D} = (1 + 4\pi \chi_e) \mathbf{E}$  and  $\kappa = \epsilon = 1 + 4\pi \chi_e$ . In MKS units  $\mathbf{D} = \epsilon(1 + \chi_e) \mathbf{E}$  and  $\kappa = \frac{\epsilon}{\epsilon_o} = 1 + \chi_e$ . Note that *for vacuum we must have*  $\chi_e = 0$ .

### 6.2.2 Magnetostatics

We proceed in analogy with electrostatics. If an external magnetic field is applied to a medium then a *magnetization field*  $\mathbf{M}$  will be set up in the medium analogous to the polarization field  $\mathbf{P}$ . The resultant magnetic field  $\mathbf{H}$  will be a linear combination of  $\mathbf{M}$  and  $\mathbf{B}$ . Thus *define* (true for *all units*)

$$\boxed{\mathbf{H} \equiv \frac{1}{\mu_o} \mathbf{B} - \lambda' \mathbf{M}}$$

(6.15)

where  $\mu_o$  and  $\lambda'$  are, as yet, undetermined constants, *both* of whose values we are free to choose. The choices are listed in Table 6.1.

We would now like to find an Ampère law for  $\mathbf{H}$ . Thus we should calculate  $\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}$  or  $\nabla \times \mathbf{H} - \alpha \frac{\partial \mathbf{D}}{\partial t}$  where  $\alpha$  is a constant, or what? Let's use the original form of Ampère's law  $\nabla \times \mathbf{B} - \frac{1}{gc^2} \frac{\partial \mathbf{E}}{\partial t}$  as our guide. This suggests taking  $\mu_o \nabla \times \mathbf{H} - \frac{1}{gc^2} \frac{1}{\epsilon_o} \frac{\partial \mathbf{D}}{\partial t}$ . Evaluating this we have

$$\begin{aligned} \mu_o \nabla \times \mathbf{H} - \frac{1}{gc^2} \frac{1}{\epsilon_o} \frac{\partial \mathbf{D}}{\partial t} &= \nabla \times \mathbf{B} - \lambda' \mu_o \nabla \times \mathbf{M} \\ &\quad - \frac{1}{gc^2} \frac{\partial \mathbf{E}}{\partial t} - \frac{\lambda}{gc^2 \epsilon_o} \frac{\partial \mathbf{P}}{\partial t} \\ &= \frac{4\pi k}{gc^2} \mathbf{j} - \lambda' \mu_o \nabla \times \mathbf{M} - \frac{4\pi k}{gc^2} \frac{\partial \mathbf{P}}{\partial t} \end{aligned}$$

where we have used equations (6.7) in the form  $4\pi k = \frac{\lambda}{\epsilon_o}$ . The term  $\nabla \times \mathbf{M}$  will correspond to bound current  $\mathbf{j}_b$  but the term  $\frac{\partial \mathbf{P}}{\partial t}$  will correspond to a new [2] "polarization" current  $\mathbf{j}_p$ . Thus we define

$$\mathbf{j} \equiv \mathbf{j}_f + \mathbf{j}_b + \mathbf{j}_p \quad (6.16)$$

to give

$$\begin{aligned} \mu_o \nabla \times \mathbf{H} - \frac{1}{gc^2} \frac{1}{\epsilon_o} \frac{\partial \mathbf{D}}{\partial t} &= \frac{4\pi k}{gc^2} \mathbf{j}_f + \left( \frac{4\pi k}{gc^2} \mathbf{j}_b - \lambda' \mu_o \nabla \times \mathbf{M} \right) \\ &\quad + \frac{4\pi k}{gc^2} \left( \mathbf{j}_p - \frac{\partial \mathbf{P}}{\partial t} \right) \end{aligned}$$

Now we have  $\nabla \times \mathbf{H}$  related to  $\mathbf{j}_f$  and  $\nabla \times \mathbf{M}$  related to  $\mathbf{j}_b$  and  $\frac{\partial \mathbf{P}}{\partial t}$  related to  $\mathbf{j}_p$ . Thus the term in parenthesis must be zero which *requires* (*all* units)

$$\boxed{\frac{\partial \mathbf{P}}{\partial t} = \mathbf{j}_p}$$

(6.17)

and (*all* units)

$$\boxed{\nabla \times \mathbf{M} = \frac{1}{\lambda' \mu_o} \frac{4\pi k}{gc^2} \mathbf{j}_b}$$



(6.18)

finally giving Ampère's law for a medium as (*all units*)

$$\boxed{\mu_o \nabla \times \mathbf{H} - \frac{1}{gc^2} \frac{1}{\epsilon_o} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi k}{gc^2} \mathbf{j}_f}$$
(6.19)

Finally we consider the magnetization of a linear medium. In electrostatics a linear medium was defined as one in which  $\mathbf{P} \equiv \epsilon_o \chi_e \mathbf{E}$  from which it followed that  $\mathbf{D} = \epsilon \mathbf{E}$  and thus also  $\mathbf{D} \propto \mathbf{P}$  and therefore any of these three relations would serve equally well as the starting point for the definition of a linear medium. Clearly the same should be true for magnetostatics, i.e.  $\mathbf{H} \propto \mathbf{B}$ ,  $\mathbf{B} \propto \mathbf{M}$  and  $\mathbf{M} \propto \mathbf{H}$ . We start with (*all units*)

$$\mathbf{M} \equiv \chi_m \mathbf{H} \quad (6.20)$$

where  $\chi_m$  is called the magnetic susceptibility. Substituting this into (6.15) yields (*for all units*)

$$\mathbf{B} = \mu_o(1 + \lambda' \chi_m) \mathbf{H} \equiv \mu \mathbf{H} \quad (6.21)$$

which serves to define the probability  $\mu$ . The *relative permeability* is defined as (*all units*)

$$\frac{\mu}{\mu_o} \equiv 1 + \lambda' \chi_m$$

### 6.2.3 Summary of Maxwell's Equations

Maxwell's equations as written in equations (31-1)-(31-4) are true *always*. They are true in material medium *and* free space. The only trick to remember is that in material medium, the charge density and current appearing on the right hand side actually are  $\rho = \rho_f + \rho_b$  and  $\mathbf{j} = \mathbf{j}_f + \mathbf{j}_b + \mathbf{j}_p$ .

We now have another set of Maxwell equations (true in *all unit systems*)

$$\frac{1}{\epsilon_o} \nabla \cdot \mathbf{D} = 4\pi k \rho_f \quad (6.22)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (6.23)$$

$$\nabla \times \mathbf{E} + g \frac{\partial \mathbf{P}}{\partial t} = 0 \quad (6.24)$$

$$\mu_o \nabla \times \mathbf{H} - \frac{1}{gc^2} \frac{1}{\epsilon_o} \frac{\partial \mathbf{D}}{\partial t} = \frac{4\pi k}{gc^2} \mathbf{j}_f \quad (6.25)$$

which again are true *always*. They are true in material media *and* free space. (However they are really only *useful* in material media.)

## 6.3 Further Dimennsional of Electrostatics

### 6.3.1 Dipoles in Electric Field

As previously discussed, when materials are placed in external electric fields they get polarized. The net polarization field  $\mathbf{P}$  is due to the way in which all the microscopic dipoles (such as polarized atoms or molecules) respond to the external field.

Thus it is of interest to us to find out what happens to a dipole in an external electri field.

The *torque on a dipole in an external electric field* (Pg. 162 of Griffiths [2]) is

$$\mathbf{M} = \mathbf{p} \times \mathbf{E}$$

(6.26)

where  $\mathbf{N}$  os the torque,  $\mathbf{E}$  is the external electric field and  $\mathbf{p}$  is the dipole moment.

If the electric field is non-uniform there will be a net force on the dipole, given by (Pg. 162 of Griffiths [2] )

$$\mathbf{F} = (\mathbf{p} \cdot \nabla) \mathbf{E}$$

(6.27)

Finally we wish to find the *energy* of a dipole in an external field  $\mathbf{E}$ . First recall that

$$\nabla(\mathbf{p} \cdot \mathbf{E}) = \mathbf{p} \times (\nabla \times \mathbf{E}) + \mathbf{E} \times (\nabla \times \mathbf{p}) + (\mathbf{p} \cdot \nabla) \mathbf{E} + (\mathbf{E} \cdot \nabla) \mathbf{p}.$$

Thus  $\nabla \times \mathbf{p} = 0$  and  $(\mathbf{E} \cdot \nabla) \mathbf{p} = 0$ . Also  $\nabla \times \mathbf{E} = 0$  to give  $\nabla(\mathbf{p} \cdot \mathbf{E}) = (\mathbf{p} \cdot \nabla) \mathbf{E}$ . Using  $\mathbf{F} = -\nabla U$  where  $U$  is the potential energy, we have

$$U = -\mathbf{p} \cdot \mathbf{E}$$

(6.28)

for the energy of dipole in an external field.

### 6.3.2 Energy Stored in a Dielectric

We previously found that the energy stored in an electric field (equivalently the work required to assemble a system of charges is

$$W = \frac{1}{8\pi k} \int_{all\ space} E^2 d\tau.$$

However for dielectric media this formula get changed to

$$W = \frac{1}{8\pi k\epsilon_o} \int_{all\ space} \mathbf{D} \cdot \mathbf{E} d\tau$$

(6.29)

(For MKS units we thus have  $W = \frac{1}{2} \int \mathbf{D} \cdot \mathbf{E} d\tau$  and for CGS units we have  $W = \frac{1}{8\pi} \int \mathbf{E} \cdot \mathbf{E} d\tau$ .)

To see how this formula comes about we consider how much work  $\delta W$  is done when we place an infinitesimal free charge  $\delta\rho_f$  into a potential  $V$ . We have

$$\delta W = \int (\delta\rho_f) V d\tau = \frac{1}{4\pi k\epsilon_o} \int [\nabla \cdot (\delta\mathbf{D})] V d\tau$$

where we have used Gauss' law  $\frac{1}{\epsilon_o} \nabla \cdot \mathbf{D} = 4\pi k\rho_f$ . Using

$$\begin{aligned} \nabla \cdot (\delta\mathbf{D}V) &= [\nabla \cdot (\delta\mathbf{D})]V + \delta\mathbf{D} \cdot \nabla V \\ &= [\nabla \cdot (\delta\mathbf{D})]V - \delta\mathbf{D} \cdot \mathbf{E} \end{aligned}$$

we have

$$\begin{aligned} \int \nabla \cdot (\delta\mathbf{D}V) d\tau &= \int [\nabla \cdot (\delta\mathbf{D})]V d\tau - \int \delta\mathbf{D} \cdot \mathbf{E} d\tau \\ &= \oint \delta\mathbf{D}V d\mathbf{A} \\ &= 0 \end{aligned}$$

because  $\mathbf{D}$  and  $V$  are zero on the boundary surface. Thus

$$\delta W = \frac{1}{4\pi k\epsilon_o} \int \delta\mathbf{D} \cdot \mathbf{E} d\tau$$

Now  $\frac{\delta(E^2)}{\delta E} = 2E$  so that  $\delta(E^2) = 2\mathbf{E} \cdot \delta\mathbf{E}$ . Writing  $\delta(\mathbf{D} \cdot \mathbf{E}) = \epsilon_0 2\mathbf{E} \cdot \delta\mathbf{E} = 2\mathbf{E} \cdot \delta\mathbf{D}$  we have  $\delta W = \frac{1}{8\pi k\epsilon_o} \int \delta(\mathbf{D} \cdot \mathbf{E}) d\tau$  to give  $W = 4 \frac{1}{8\pi k\epsilon_o} \int \mathbf{D} \cdot \mathbf{E} d\tau$  as in equation (6.29).

### 6.3.3 Potential of a Polarized Dielectric

In chapter 4 we found that the dipole term in the general multipole expression of the potential was

$$V = k \frac{\mathbf{p} \cdot \hat{\mathbf{R}}}{R^2}$$

where  $\mathbf{p}$  is the dipole moment which we defined as  $\mathbf{p} \equiv \int \mathbf{P} d\tau$  and

$$V = k \int \frac{\mathbf{P} \cdot \hat{\mathbf{R}}}{R^2} \quad (6.30)$$

and using  $\nabla(\frac{1}{R})$  this becomes

$$V = k \int \mathbf{P} \cdot \nabla\left(\frac{1}{R}\right) d\tau = k \int \nabla \cdot \left(\frac{1}{R} \mathbf{P}\right) d\tau - k \int \frac{1}{R} (\nabla \cdot \mathbf{P}) d\tau$$

to give

$$V = k \int \frac{1}{R} \mathbf{P} \cdot d\mathbf{a} - k \int \frac{1}{R} (\nabla \cdot \mathbf{P}) d\tau \quad (6.31)$$

Now recall that  $V = k \frac{q}{R}$  for a point charge and  $V = k \int \frac{\rho d\tau}{R}$  for a volume charge and  $V = k \int \frac{\sigma da}{R}$  for a surface charge. Thus the first term in (6.31) looks like the potential of a surface charge  $\sigma_b \equiv \mathbf{P} \cdot \mathbf{n}$ , so that  $\mathbf{P} \cdot d\mathbf{a} = \sigma_b da$  and the second term looks like the potential of a volume charge  $\rho_b = -\nabla \cdot \mathbf{P}$  (see equation 6.10). Thus we have

$$\boxed{V = k \int \frac{\sigma_b da}{R} + k \int \frac{\rho_b d\tau}{R}}$$

(6.32)

where

$$\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}} \quad (6.33)$$

and

$$\nabla \cdot \mathbf{P} = -\rho_b.$$

Equation (6.32) means that the potential (and field) of a polarized object is the same as that produced by a volume charge density  $\rho_b = -\nabla \cdot \mathbf{P}$  plus a surface charge density  $\sigma_b \equiv \mathbf{P} \cdot \hat{\mathbf{n}}$ .



## Chapter 7

# ELECTRODYNAMICS AND MAGNETODYNAMICS

In electrostatics and magnetostatics we can generally treat the electric and magnetic fields as separate entities. However when we come to time-dependent problems, we need to consider combined electromagnetic fields.

### 7.0.4 Faradays's Law of Induction

"The discovery in 1820 that there was a close connection between electricity and magnetism was very exciting until then, the two subjects had been considered as quite independent. The first discovery was that currents in wires make magnetic fields; then, in the same year, it was found that wires carrying currents in a magnetic field have forces on them. One of the excitements whenever there is a mechanical force is the probability of using it in an engine to do work. Almost immediately after their discovery, people started to design electric motors using the forces on current-carryin wires" (quoted from Feynman [?], pg.16-1)

Thus physicists found that current  $i$  makes magnetic field  $\mathbf{B}$ . The question was does  $\mathbf{B}$  make  $\mathbf{E}$ ? Magnetic field cannot move static charges, but electric field can. (Recall  $\mathbf{F} = q(\mathbf{E} + g\mathbf{v} \times \mathbf{B})$ ). That is if  $\mathbf{B}$  makes  $\mathbf{E}$  it will make current  $i$  flow. Does  $\mathbf{B}$  make  $i$ ? People tried various things to find this our. For instance they put a large magnet next to a wire (to try to get current to flow) and also put two close wires parallel but no effects were

observed.

In 1840 Faraday discovered the essential feature that had been missed. Electric effects only exist when there is something it changing. Thus if a magnet is *moved* near a circuit then a current will flow. If one pair of close, parallel wires has a *changing* current, then a current is induced in the other. Thus we say that currents are *induced* by *changing magnetic field*.

However it *cannot* be the magnetic field which directly causes the current to flow because negative fields produce no forces as stationary charges. The only thing that can cause a stationary charge to move is an electric field. Thus the changing magnetic field must somehow be producing a corresponding *electric* field.

These effects are embodied in Faraday's law

$$\nabla \times \mathbf{E} + g \frac{\partial \mathbf{B}}{\partial t} = 0$$

where the constant  $g$  relates the units (whatever they are chosen to be) of the two previously unrelated quantities  $\mathbf{E}$  and  $\mathbf{B}$ . In integral form Faraday's law is

$$\oint \mathbf{E} \cdot d\ell + g \frac{\partial \phi_g}{\partial \tau} = 0$$

The emf (electromotive force, which is *not* actually a force) is defined as

$$\mathcal{E} \equiv \oint \mathbf{E} \cdot d\ell \tag{7.1}$$

so that Faraday's law is

$$\boxed{\mathcal{E} = -g \frac{\partial \phi_g}{\partial t}} \tag{7.2}$$

which says that the emf in a circuit is due to a changing magnetic flux.

*The emf, as defined in (7.1) is actually the tangential force (due to dot product  $\mathbf{E} \cdot d\ell$ ) per unit charge, integrated over is just a field  $\mathbf{E}$ .) If you have an emf  $\mathcal{E}$  in a circuit then it means that you get current flow according to Ohm's law  $\mathcal{E} = iR$ . Batteries or voltage sources do the same thing.*

The emf is also sometimes defined as the work done, per unit charge, by an energy source in a circuit. Thus

$$\mathcal{E} \equiv \frac{W}{q} \frac{\oint \mathbf{F} \cdot d\ell}{q} = \oint \mathbf{E} \cdot d\ell \quad (7.3)$$

Actually Faraday's complete discovery was that emfs or currents can be generated in a wire in 3 different ways; 1) by *moving the wire* near an adjacent circuit, 2) by *changing the current* in an adjacent circuit and 3) by moving a magnet near the wire. All 3 situations agree with the statement that emf or current is due to a changing magnetic flux linked by the circuit as specified in Faraday's law (7.2).

*It was Faraday's observations and experiments that lead him to propose his law in the form of equation (7.2).* This states that the emf (or current) is proportional to the rate of change of the magnetic flux. One might therefore think that the constant  $g$  is a number to be determined from experiment, but actually it just depends on the unit system employed. Figure 7.1 gives a better idea of what is meant by the flux linking the circuit.

The minus sign in Faraday's law (7.2) is consistent (comes from) *Lenz's law* which states that the direction of the induced currents (and *its* accompanying magnetic flux) is always such as to oppose the change of flux through the circuit. This is illustrated in Figure 7.2. Note that the direction of the current flow, specified by Lenz's law, is really nothing more than an application of  $\mathbf{F} = gq\mathbf{v} \times \mathbf{B}$ .

Let us illustrate how Lenz's law works for the current induced in a long, straight wire segment of a circuit loop in the following example.

**Example 7.1.1.** Let  $\mathbf{B}'$  be the magnetic field *inside* a current loop which is produced by an induced current of the current loop within an external magnetic field  $\mathbf{B}$  as shown in Figure 7.2. Show that  $\mathbf{B}'$  and  $\mathbf{B}$  have opposite signs if the area increases and have like signs if the area decreases.

**Solution** In example 5.3.1 we found that magnetic field due to a current in a long, straight wire as

$$B' = \frac{2k_m i}{d}$$

where  $d$  is the distance from the wire. Using Ohm's law

$$\mathcal{E} = iR$$



we have

$$B' = \frac{2k_m}{d} \frac{\mathcal{E}}{R} = \frac{-2k_m}{dR} g \frac{\partial \phi_B}{\partial t}$$

where we have used (7.2). Writing  $\phi_B = Ba$  where  $a$  is the area of the loop inside the external flux  $\mathbf{B}$ , we have

$$B' = -\frac{2k_m}{dR} gB \frac{da}{dt}$$

where  $\frac{da}{dt}$  is the rate of change of the area. Clearly if  $\frac{da}{dt}$  is positive (area increasing) then  $B' \propto -B$  meaning that  $\mathbf{B}'$  and  $\mathbf{B}$  are in the same direction ( $B' \propto +B$ ).

Faradays law (7.2) can also be written

$$\mathcal{E} = \oint \mathbf{E} \cdot d\boldsymbol{\ell} = -g \frac{\partial}{\partial t} \int \mathbf{B} \cdot d\mathbf{a} \quad (7.3)$$

which clearly shows that flux can be changed by changing  $\mathbf{B}$  or by changing the shape or position or orientation of the circuit. Equation (7.3) is a far reaching generalization of Faraday's law (7.2).  $\oint d\boldsymbol{\ell}$  needn't be over a closed electric circuit but can simply be any closed geometrical path in space. Thus this form of Faraday's law (7.3) gives a relation between  $\mathbf{E}$  and  $\mathbf{B}$  themselves and suggests a unification of electric and magnetic phenomena.

### 7.0.5 Analogy between Faraday field and Magnetostatics

( A good reference for this section is pg. 287-289, including Examples 6 and 7 from Griffiths [2]).

A glance at the Maxwell equations (3.1 to 3.4) reveals that electric field  $\mathbf{E}$  can arise from two *very* different sources. Gauss's law (3.23) tells us that  $\mathbf{E}$  can arise from a charge  $q$ . while Faraday's law (3.25) tells us that  $\mathbf{E}$  can arise from a changing magnetic flux. Similarly Ampère's law (3.26) tells us that magnetic field  $\mathbf{B}$  can arise from a current density  $\mathbf{j}$  or from a changing electric flux.

For the *static* case, Ampère's law tells us that magnetic field only arises due to a current density

$$\nabla \times \mathbf{B} = 4\pi k_m \mathbf{j} \quad (7.4)$$

while it is always true that

$$\nabla \cdot \mathbf{B} = 0 \quad (7.5)$$

Faraday's law (non-static) is written

$$\nabla \times \mathbf{E} = -g \frac{\partial \mathbf{B}}{\partial t} \quad (7.6)$$

Now if the electric field *only* arises from the changing magnetic flux and if no change are present then

$$\nabla \cdot \mathbf{E} = 0 \quad (7.7)$$

also. Thus we have an exact analogy between the magnetic field due to a static charge density and the electric field due to  $\frac{\partial \mathbf{B}}{\partial t}$ . The integral forms are

$$\oint \mathbf{B} \cdot d\ell = 4\pi k_m i \quad (7.8)$$

and

$$\oint \mathbf{E} \cdot d\ell = -g \frac{\partial \phi_B}{\partial t} \quad (7.9)$$

Thus "Faraday-induced electric fields are determined by  $-\frac{\partial \mathbf{B}}{\partial t}$  is exactly the same way as magnetostatic fields are determined by  $\mathbf{j}$ ." (Griffith pg.287)

The Biot-Savart law

$$\mathbf{B} = k_m \int \frac{\mathbf{j} \times \hat{\mathbf{R}} R^2 d\tau}{r^3} \quad (7.10)$$

translates into

$$\mathbf{E} = -\frac{g}{4\pi} \int \frac{\frac{\partial \mathbf{B}}{\partial t} \times \hat{\mathbf{R}}}{R^2} d\tau = \frac{d}{dt} \left[ \frac{g}{4\pi} \int \frac{\mathbf{B} \times \hat{\mathbf{R}}}{R^2} d\tau \right] \quad (7.11)$$

If there is symmetry in the problem we use the techniques of Ampèrian loops in (7.9) just as we did for (7.8).

(Students study examples 6 and 7 on pg. 288-289 of Griffiths.)

## 7.1 Ohm's Law and Electrostatic Force

Introductory physics textbooks usually write Ohm's law as

$$\boxed{V = iR}$$

(7.12)

where  $V$  is the potential (or voltage) and  $i$  is the current flowing through a resistor  $R$ . However books on electromagnetic theory [10, 2] write Ohm's law as

$$\mathbf{j} = \sigma \mathbf{E} \quad (7.13)$$

where  $\mathbf{j}$  is the current density,  $\sigma$  is the conductivity and  $\mathbf{E}$  is the electric field. Let us therefore discuss Ohm's law from first principles.

Recall the Lorentz force law

$$\mathbf{F} = q(\mathbf{E} + g\mathbf{v} \times \mathbf{B}) \quad (7.14)$$

which gives the electromagnetic force on a charge  $q$ . Obviously the *force per unit charge* defined as  $\mathbf{f} \equiv \frac{\mathbf{F}}{q}$  is

$$\mathbf{f} = \mathbf{E} + g\mathbf{v} \times \mathbf{B} \quad (7.15)$$

Obviously if  $\mathbf{v}$  or  $\mathbf{B}$  are zero then the force per unit charge is nothing more than the electric  $\mathbf{f} = \mathbf{E}$ , but for a moving charge in a magnetic field we have the more general relation (7.15).

*Experimentally it is found that the current flow in a circuit is proportional to the force per unit charge.* If you think about it this makes perfect sense. Actually it is the current density  $\mathbf{j}$  which is proportional to  $\mathbf{f}$  and the proportionality constant is called the conductivity  $\sigma$ . (The resistivity is defined as  $R \equiv \frac{1}{\sigma}$ .) Thus Ohm's law is

$$\boxed{\mathbf{j} = \sigma \mathbf{f}} \quad (7.16)$$

This represents Ohm's law in its most general form. Two things are worth noting. Firstly Ohm's law is an experimental observation. Secondly Ohm's law should be regarded just like Hooke's law  $\mathbf{F} = -kx$  for a spring. We all know that Hooke's law is only valid (experimentally) for small oscillations and similarly Ohm's law is only valid for small electrical forces. Just as Hooke's law has non-linear corrections for large displacements, so too does Ohm's law have non-linear corrections for large currents. Equation (7.16) yields  $\mathbf{j} = \sigma(\mathbf{E} + g\mathbf{v} \times \mathbf{B})$ . In circuits usually  $\mathbf{v}$  and  $\mathbf{B}$  are quite small, so that the approximate version is  $\mathbf{j} \approx \sigma \mathbf{E}$  which is (7.13).

Imagine we have a cylindrical wire of cross sectioned area  $A$ , length  $\ell$  with an electric field  $\mathbf{E}$  applied in the longitudinal direction as shown in figure 7.3.

Assume that there is a voltage  $V$  between the two ends of the wire. Using  $V(p) = -\int_{\infty}^p \mathbf{E} \cdot d\ell$ , it is evident that for a *uniform* electric field,  $\mathbf{E}$  can be taken outside the integral to give  $E = \frac{V}{\ell}$ . The current therefore is

$$i = jA \approx \sigma EA = \sigma \frac{V}{\ell} A \quad (7.17)$$

showing that the voltage is proportional to the current  $V \propto i$ . The *resistance*  $R$  is the proportionality constant giving  $V = iR$ , as given in equation (7.12). For our example the resistance is

$$R = \frac{\ell}{\sigma A} \equiv R \frac{\ell}{A} \quad (7.18)$$

which, again if you think about it, makes perfect sense.

Just from looking at units, the power  $P$  dissipated across a resistor is

$$P = Vi = i^2 R \quad (7.19)$$

but the more general case uses the proper definition of power as  $P \equiv \oint \mathbf{f} \cdot \mathbf{v}$  which gives

$$P = \mathbf{F} \cdot \mathbf{v} = \int \rho b f \mathbf{E} \cdot \mathbf{v} d\tau = \int \mathbf{E} \cdot \mathbf{j} d\tau = \sigma \int E^2 d\tau \quad (7.20)$$

(NNN see Example 1, Pg.271 of Griffith)

Ohm's law can be related to the electromotive force  $\mathcal{E}$ . We defined this before as  $\mathcal{E} \equiv \oint \mathbf{E} \cdot d\ell$ . The electric field  $\mathbf{E}$  is the force per unit charge in the absence of  $\mathbf{v}$  or  $\mathbf{B}$ . Thus a more general definition would be  $\mathcal{E} \equiv \oint \mathbf{f} \cdot d\ell$  which gives

$$\mathcal{E} \equiv \oint \mathbf{f} \cdot d\ell = \oint \frac{bfj}{\sigma} \cdot d\ell = \oint \frac{i}{A\sigma} d\ell = i \oint \frac{d\ell}{A\sigma} = iR. \quad (7.21)$$

Thus the emf also obeys the form of Ohm's law given in (7.12).

## Chapter 11

# **ELECTROMAGNETIC WAVES**



## Chapter 12

# SPECIAL RELATIVITY





# Bibliography

- [1] H.A. Atwater, *Introduction to General Relativity*, (Pergamon, New York, 1974).
- [2] D.J. Griffiths, *Introduction to Electrodynamics*, (Prentice-Hall, Englewood Cliffs, New Jersey, 1989). **text book for this course**. QC680.G74
- [3] J.T. Cushing, *Applied Analytical Mathematics for Physical Scientists*, (Wiley, New York, 1975).
- [4] J.B. Fraleigh and R.A. Beauregard, *Linear Algebra*, (Addison-Wesley, Reading, Massachusetts, 1987).
- [5] E.J.Purcell and D. Varberg, *Calculus and Analytic Geometry*, 5th ed., (Prentice-Hall, Englewood Cliffs, New Jersey, 1987). QA303.P99
- [6] R.A. Serway, *Physics for Scientists and Engineers*, 3rd ed. (Saunders, Philadelphia, 1990). QC23.S46
- [7] F.W. Byron and Fuller, *Mathematics of Classical and Quantum Physics*, vols. 1 and 2, (Addison-Wesley, Reading, Massachusetts, 1969). QC20.B9
- [8] G.B. Arfken and H.J. Weber, *Mathematical Methods for Physicists*, 4th ed., (Academic Press, San Diego, 1995). QA37.2.A74
- [9] H.C. Ohanian, *Classical Electrodynamics*, (Allyn and Bacon, Boston, 1988). QC631.O43
- [10] J.D. Jackson, *Classical Electrodynamics*, (Wiley, New York, 1975). QC631.J3

- [11] J.B. Marion, *Classical Electromagnetic Radiation*, (Academic Press, New York, 1965). QC631.M37
- [12] G.R. Fowles and G.L. Cassiday, *Analytical Mechanics*, 5th ed. (Saunders College Publishing, New York, 1993).
- [13] J.B. Marion, *Classical Dynamics of Particles and Systems*, 3rd ed., (Harcourt, Brace, Jovanovich College Publishers, New York, 1988). QA845 .M38
- [14] J. Gleick, *Chaos*, (Viking, New York, 1987). Q172.5.C45
- [15] F. Mandl and x. Shaw, *Quantum field theory*, (Wiley, New York, 1984). QC174.45.M32
- [16] M. Shamos, *Great experiments in physics*, (Holt, New York, 1984). QC7.S47
- [17] F. Halzen and A. Martin, *Quarks and leptons*, (Wiley, New York, 1984). QC793.5.Q2522 H34